# Simple Type Theory as Framework for Combining Logics

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synonyms in this talk

Church's Simple Type Theory

Classical Higher Order Logic (HOL)

- simple types $\alpha, \beta ::= \iota | o | \alpha \to \beta$ (opt. further base types)
- HOL defined by

$$
\begin{aligned}
s, t \quad ::= \quad & p_\alpha \mid X_\alpha \\
& \mid (\lambda X_\alpha \text{.} s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid (\Pi_{(\alpha\to o)\to o}\, t_{\alpha\to o})_o \\
& \mid (s_\alpha =_{\alpha\to\alpha\to o} t_\alpha)_o
\end{aligned}
$$

- HOL is well understood
  - Origin (Church, J.Symb.Log., 1940)
  - Henkin semantics (Henkin, J.Symb.Log., 1950)
    (Andrews, J.Symb.Log., 1972)
  - Extens./Intens. (BenzmüllerEtAl., J.Symb.Log., 2004)
    (Muskens, J.Symb.Log., 2007)

# Opinions about HOL:

- HOL is expressive

but . . .

- HOL can not be effectively automated
- HOL is a classical logic and not easily compatible with
  - modal logics
  - intuitionistic logic
  - . . .
- HOL can not fruitfully serve as a basis for combining logics

# Opinions about HOL: Position of this talk

- HOL is expressive and we exploit this here

but ...

- HOL can ~~not~~ be effectively automated (at least partly)
- HOL is a classical logic and ~~not~~ easily compatible with
  - (normal) modal logics
  - intuitionistic logic
  - ...
- HOL can ~~not~~ fruitfully serve as a basis for combining logics

... I will give theoretical and practical evidence

Quantified Multimodal Logics (QML)
as HOL Fragments
(jww Larry Paulson)

# Quantified Multimodal Logics (QML)

- QML defined by

$$\begin{aligned} s,t \quad ::= \quad & P \mid (k\,X^1 \ldots X^n) \\ & \mid \neg\, s \mid s \vee t \\ & \mid \Box_r\, s \\ & \mid \forall X_\blacksquare s \mid \forall P_\blacksquare s \end{aligned}$$

- Kripke style semantics
  - different notions of models: (Fitting, J.Symb.Log., 2005)
    **QS5**$\pi^-$
    **QS5**$\pi$
    **QS5**$\pi^+$

# Quantified Multimodal Logics (QML)

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\end{aligned}
$$

- Kripke style semantics
  - different notions of models: (Fitting, J.Symb.Log., 2005)
    $\mathbf{QS5}\pi^- \longrightarrow \mathbf{QK}\pi^-$
    $\mathbf{QS5}\pi \longrightarrow \mathbf{QK}\pi$ (correspondence to Henkin models)
    $\mathbf{QS5}\pi^+ \longrightarrow \mathbf{QK}\pi^+$

# (Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

## Straightforward Encoding

- base type $\iota$: non-empty set of possible worlds
- base type $\mu$: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \to o$

## QML Operators as abbreviations for specific HOL terms

$$\neg = \lambda\phi.\lambda W\iota.\neg\phi\, W$$

$$\vee = \lambda\phi.\lambda\psi.\lambda W.\phi\, W \vee \psi\, W$$

$$\Box = \lambda R.\lambda\phi.\lambda W.\forall V.\neg R\, W\, V \vee \phi\, V$$

$$\Pi^{\mu} = \lambda\tau.\lambda W.\forall X.(\tau\, X)\, W \qquad \text{(quantif. over individuals)}$$

$$\Pi^{\iota\to o} = \lambda\tau.\lambda W.\forall P.(\tau\, P)\, W \qquad \text{(quantif. over propositions)}$$

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## QML Operators as abbreviations for specific HOL terms

$$\neg = \lambda \phi_{\iota\to o} \lambda W_{\iota} \neg \phi\, W$$

$$\vee = \lambda \phi_{\iota\to o} \lambda \psi_{\iota\to o} \lambda W_{\iota} \phi\, W \vee \psi\, W$$

$$\Box = \lambda R_{\iota\to\iota\to o} \lambda \phi_{\iota\to o} \lambda W_{\iota} \forall V_{\iota} \neg R\, W\, V \vee \phi\, V$$

$$\Pi^{\mu} = \lambda \tau_{\mu\to(\iota\to o)} \lambda W_{\iota} \forall X_{\mu} (\tau\, X)\, W$$

$$\Pi^{\iota\to o} = \lambda \tau_{(\iota\to o)\to(\iota\to o)} \lambda W_{\iota} \forall P_{\iota\to o} (\tau\, P)\, W$$

# (Normal) QML as Fragment of HOL

Encoding of validity

$$\text{valid} \quad = \quad \lambda\phi_{\iota\to o}.\forall W_\iota.\phi\, W$$

# Example: In all $r$-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \, \exists P_{\iota\to o}. \, P$$

then automatically rewrite abbreviations

$$\begin{aligned}
\Box_r &\xrightarrow{\text{rewrite}} \dots \\
\exists &\xrightarrow{\text{rewrite}} \dots \\
\text{valid} &\xrightarrow{\text{rewrite}} \dots \\
&\xrightarrow{\beta\eta\downarrow} \forall W_\iota. \forall Y_\iota. \neg r \, W \, Y \vee (\neg \forall P_{\iota\to o}. \neg (P \, Y))
\end{aligned}$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ...
here the provers need to generate witness term $P = \lambda Y_\iota. \top$)

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\text{valid} & \xrightarrow{\text{rewrite}} & \ldots \\
 & \xrightarrow{\beta\eta\downarrow} & \forall W_\iota\text{.}\forall Y_\iota\text{.}\neg r\ W\ Y \vee (\neg\forall P_{\iota\to o}\text{.}\neg(P\ Y))
\end{array}$$

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# Example: In all $r$-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \, \exists P_{\iota\to o^{\blacksquare}} \, P$$

then automatically rewrite abbreviations

$$\begin{array}{rcl} \Box_r & \xrightarrow{\text{rewrite}} & \ldots \\ \exists & \xrightarrow{\text{rewrite}} & \ldots \\ \text{valid} & \xrightarrow{\text{rewrite}} & \ldots \\ & \xrightarrow{\beta\eta\downarrow} & \forall W_{\iota^{\blacksquare}} \forall Y_{\iota^{\blacksquare}} \neg r\, W\, Y \vee (\neg \forall P_{\iota\to o^{\blacksquare}} \neg (P\, Y)) \end{array}$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ...
here the provers need to generate witness term $P = \lambda Y_{\iota^{\blacksquare}} \top$)

# Soundness and Completeness

Theorem:

$$\models^{QML}_{\mathbf{QK}\pi} s \quad \text{if and only if} \quad \models^{HOL}_{Henkin} \text{valid } s_{\iota\to o}$$

(BenzmüllerPaulson, Techn.Report, 2009)

# Further interesting Fragments of HOL

- Intuitionistic Logic
  (exploiting Gödel's translation to S4)
  (BenzmüllerPaulson, Log.J.IGPL, 2010)
- Access Control Logics
  (exploiting a translation by Garg and Abadi)
  (Benzmüller, SEC, 2009)
- Region Connection Calculus — later in this talk
- ...

# Reasoning *about* Combinations of Logics

# Reasoning *about* Combinations of Logics: Correspondence

Correspondences between properties of accessibility relations

$$\begin{aligned} \text{symmetric} &= \lambda R.\forall S, T.R\,S\,T \Rightarrow R\,T\,S \\ \text{serial} &= \lambda R.\forall S.\exists T.R\,S\,T \end{aligned}$$

and corresponding axioms

$$\forall R.\text{symmetric}\,R \overset{0.0s}{\Leftarrow}$$

$$\overset{0.0s}{\Rightarrow} \quad \text{valid}\ \forall \phi.\phi \supset \Box_R \Diamond_R \phi$$

$$\forall R.\text{serial}\,R \overset{0.0s}{\Leftarrow}$$

$$\overset{0.0s}{\Rightarrow} \quad \text{valid}\ \forall \phi.\Box_R \phi \supset \Diamond_R \phi$$

Such proofs can be automated with LEO-II in no-time!

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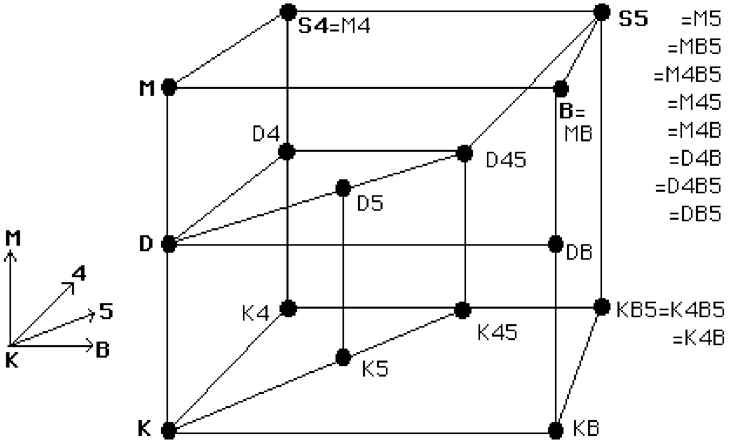
$$\forall R.\text{serial}\ R \overset{0.0s}{\Leftarrow}$$

$$\overset{0.0s}{\Rightarrow} \quad \text{valid}\ \forall \phi.\Box_R \phi \supset \Diamond_R \phi$$

Such proofs can be automated with LEO-II in no-time!

# Reasoning *about* Combinations of Logics: Cube




(cf. John Halleck's website)

$$\forall R.$$

$$\left.\begin{array}{ll} & \text{valid } \forall\phi.\, \Box_R\,\phi \supset \phi \\ \wedge & \text{valid } \forall\phi.\, \Diamond_R\,\phi \supset \Box_R \Diamond_R\,\phi \end{array}\right\} = M5$$

$$\Leftrightarrow$$

$$\left.\begin{array}{ll} & \text{valid } \forall\phi.\, \Box_R\,\phi \supset \Diamond_R\,\phi \\ \wedge & \text{valid } \forall\phi.\, \Box_R\,\phi \supset \Box_R \Box_R\,\phi \\ \wedge & \text{valid } \forall\phi.\, \phi \supset \Box_R \Diamond_R\,\phi \end{array}\right\} = D4B$$

$$\forall R.\ \left.\begin{array}{ll} & \text{valid}\ \forall\phi.\ \Box_R\,\phi \supset \phi \\ \wedge & \text{valid}\ \forall\phi.\ \Diamond_R\,\phi \supset \Box_R \Diamond_R\,\phi \end{array}\right\} = M5$$

$$\Leftrightarrow$$

$$\left.\begin{array}{ll} & \text{serial}\ R \\ \wedge & \text{valid}\ \forall\phi.\ \Box_R\,\phi \supset \Box_R \Box_R\,\phi \\ \wedge & \text{symmetric}\ R \end{array}\right\} = D4B$$

$$\forall R.\quad \left.\begin{array}{ll} & \text{reflexive } R \\ \wedge & \text{euclidean } R \end{array}\right\} = M5$$

$$\Leftrightarrow$$

$$\left.\begin{array}{ll} & \text{serial } R \\ \wedge & \text{transitive } R \\ \wedge & \text{symmetric } R \end{array}\right\} = D4B$$

$$\forall R.$$

$$\left.\begin{array}{ll} & \text{reflexive } R \\ \wedge & \text{euclidean } R \end{array}\right\} = M5$$

$$\overset{0.1s}{\Leftrightarrow}$$

$$\left.\begin{array}{ll} & \text{serial } R \\ \wedge & \text{transitive } R \\ \wedge & \text{symmetric } R \end{array}\right\} = D4B$$

Proof with LEO-II in 0.1s

# Reasoning *about* Combinations of Logics: Logics Cube

$$
\begin{aligned}
S5 = M5 &\Leftrightarrow MB5 \\
&\Leftrightarrow M4B5 \\
&\Leftrightarrow M45 \\
&\Leftrightarrow M4B \\
&\Leftrightarrow D4B \\
&\Leftrightarrow D4B5 \\
&\Leftrightarrow DB5
\end{aligned}
$$

$$
\begin{aligned}
KB5 &\Leftrightarrow K4B5 \\
&\Leftrightarrow K4B
\end{aligned}
$$

$$
\begin{aligned}
M5 &\Rightarrow D45 \\
D45 &\Rightarrow M5
\end{aligned}
$$

# Reasoning *about* Combinations of Logics: Logics Cube

$$
\begin{array}{lcl}
S5 = M5 & \overset{0.1s}{\Leftrightarrow} & MB5 \\
 & \overset{0.2s}{\Leftrightarrow} & M4B5 \\
 & \overset{0.1s}{\Leftrightarrow} & M45 \\
 & \overset{0.1s}{\Leftrightarrow} & M4B \\
 & \overset{0.1s}{\Leftrightarrow} & D4B \\
 & \overset{0.2s}{\Leftrightarrow} & D4B5 \\
 & \overset{0.1s}{\Leftrightarrow} & DB5
\end{array}
$$

$$
\begin{array}{lcl}
KB5 & \overset{0.2s}{\Leftrightarrow} & K4B5 \\
 & \overset{0.1s}{\Leftrightarrow} & K4B
\end{array}
$$

$$
\begin{array}{lcl}
M5 & \overset{0.1s}{\Rightarrow} & D45 \\
D45 & \overset{???}{\Rightarrow} & M5
\end{array}
$$

Proofs with LEO-II $< 0.2s$

# Reasoning *about* Combinations of Logics: Logics Cube

$$
\begin{array}{rcl}
S5 = M5 & \Leftrightarrow & MB5 \\
 & \Leftrightarrow & M4B5 \\
 & \Leftrightarrow & M45 \\
 & \Leftrightarrow & M4B \\
 & \Leftrightarrow & D4B \\
 & \Leftrightarrow & D4B5 \\
 & \Leftrightarrow & DB5
\end{array}
$$

$$
\begin{array}{rcl}
KB5 & \Leftrightarrow & K4B5 \\
 & \Leftrightarrow & K4B
\end{array}
$$

$$
\begin{array}{rcl}
M5 & \Rightarrow & D45 \\
D45 & \not\Rightarrow & M5
\end{array}
$$

Countermodel 34.4s (IsabelleN)

# Reasoning *about* Combinations of Logics: Segerberg

(Segerberg, 1973) discusses a 2-dimensional logic providing two S5 modalities $\Box_a$ and $\Box_b$. He adds further axioms stating that these modalities are commutative and orthogonal. It actually turns out that orthogonality is already implied in this context.

$$
\begin{array}{ll}
 & \text{reflexive } a, \text{transitive } a, \text{euclid. } a, \\
 & \text{reflexive } b, \text{transitive } b, \text{euclid. } b, \\
 & \text{valid } \forall\phi.\ \Box_a \Box_b \phi \Leftrightarrow \Box_b \Box_a \phi \\
\vdash^{HOL} & \\
 & \text{valid } \forall\phi,\psi.\ \Box_a (\Box_a \phi \vee \Box_b \psi) \supset (\Box_a \phi \vee \Box_a \psi) \\
 & \wedge \\
 & \text{valid } \forall\phi,\psi.\ \Box_b (\Box_a \phi \vee \Box_b \psi) \supset (\Box_b \phi \vee \Box_b \psi)
\end{array}
$$

# Reasoning *about* Combinations of Logics: Segerberg

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$$\vdash^{HOL} \begin{array}{l} \text{reflexive } a, \text{transitive } a, \text{euclid. } a, \\ \text{reflexive } b, \text{transitive } b, \text{euclid. } b, \\ \text{valid } \forall \phi_{\bullet}\ \Box_a \Box_b \phi \Leftrightarrow \Box_b \Box_a \phi \\ \qquad\qquad\qquad \text{proof by LEO-II in 0.2s} \\ \text{valid } \forall \phi, \psi_{\bullet}\ \Box_a (\Box_a \phi \vee \Box_b \psi) \supset (\Box_a \phi \vee \Box_a \psi) \\ \wedge \\ \text{valid } \forall \phi, \psi_{\bullet}\ \Box_b (\Box_a \phi \vee \Box_b \psi) \supset (\Box_b \phi \vee \Box_b \psi) \end{array}$$

Reasoning *within* Combined Logics

## Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

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(adapting (Baldoni, PhD, 1998))

- epistemic modalities:

  $\Box_a, \Box_b, \Box_c$: three wise men

  $\Box_{fool}$: common knowledge

- predicate constant:

  $ws$: 'has white spot'

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(adapting (Baldoni, PhD, 1998))

- common knowledge:
  at least one of the wise men has a white spot

$$\text{valid } \Box_{\text{fool}} (ws\ a) \vee (ws\ b) \vee (ws\ c)$$

if X one has a white spot then Y can see this

$$(\text{valid } \Box_{\text{fool}} (ws\ X) \Rightarrow \Box_Y (ws\ X))$$

if X has not a white spot then Y can see this

$$\text{valid } \Box_{\text{fool}} \neg (ws\ X) \Rightarrow \Box_Y \neg (ws\ X))$$

$$X \neq Y \in \{a, b, c\}$$

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(adapting (Baldoni, PhD, 1998))

- if X knows $\phi$ then Y knows that he knows $\phi$

$$\text{valid}\,\forall\phi_{\blacksquare}\,(\Box_X\,\phi \Rightarrow \Box_Y\,\Box_X\,\phi)$$

- if X does not know $\phi$ then Y does not know $\phi$

$$\text{valid}\,\forall\phi_{\blacksquare}\,(\neg\,\Box_X\,\phi \Rightarrow \Box_Y\,\neg\,\Box_X\,\phi)$$

$$X \neq Y \in \{a, b, c\}$$

- axioms for common knowledge

$$\text{valid}\,\forall\phi_{\blacksquare}\,\Box_{fool}\,\phi \Rightarrow \phi$$
$$\text{valid}\,\forall\phi_{\blacksquare}\,\Box_{fool}\,\phi \Rightarrow \Box_{fool}\,\Box_{fool}\,\phi$$
$$\forall R_{\blacksquare}\,\text{valid}\,\forall\phi_{\blacksquare}\,\Box_{fool}\,\phi \Rightarrow \Box_R\,\phi$$

## Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapting (Baldoni, PhD, 1998))

- a, b do not know that they have a white spot

$$\text{valid}\ \neg\,\Box_a\,(ws\ a) \qquad \text{valid}\ \neg\,\Box_b\,(ws\ b)$$

- prove that c does know he has a white spot:

$$\ldots \vdash^{HOL} \text{valid}\ \Box_c\,(ws\ c)$$

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$$\text{valid}\ \neg\,\Box_a\,(ws\ a) \qquad\qquad \text{valid}\ \neg\,\Box_b\,(ws\ b)$$

- prove that c does know he has a white spot:

$$\ldots \vdash^{HOL} \text{valid}\ \Box_c\,(ws\ c)$$

LEO-II can prove this result in 0.3s

Region Connection Calculus (RCC) (RandellCuiCohn, 1992)
as fragment of HOL:

| | | |
|---|---|---|
| disconnected : | $dc$ | $= \lambda X_\tau . \lambda Y_\tau . \neg(c\ X\ Y)$ |
| part of : | $p$ | $= \lambda X_\tau . \lambda Y_\tau . \forall Z . ((c\ Z\ X) \Rightarrow (c\ Z\ Y))$ |
| identical with : | $eq$ | $= \lambda X_\tau . \lambda Y_\tau . ((p\ X\ Y) \wedge (p\ Y\ X))$ |
| overlaps : | $o$ | $= \lambda X_\tau . \lambda Y_\tau . \exists Z . ((p\ Z\ X) \wedge (p\ Z\ Y))$ |
| partially o : | $po$ | $= \lambda X_\tau . \lambda Y_\tau . ((o\ X\ Y) \wedge \neg(p\ X\ Y) \wedge \neg(p\ Y\ X))$ |
| ext. connected : | $ec$ | $= \lambda X_\tau . \lambda Y_\tau . ((c\ X\ Y) \wedge \neg(o\ X\ Y))$ |
| proper part : | $pp$ | $= \lambda X_\tau . \lambda Y_\tau . ((p\ X\ Y) \wedge \neg(p\ Y\ X))$ |
| tangential pp : | $tpp$ | $= \lambda X_\tau . \lambda Y_\tau . ((pp\ X\ Y) \wedge \exists Z . ((ec\ Z\ X) \wedge (ec\ Z\ Y)))$ |
| nontang. pp : | $ntpp$ | $= \lambda X_\tau . \lambda Y_\tau . ((pp\ X\ Y) \wedge \neg\exists Z . ((ec\ Z\ X) \wedge (ec\ Z\ Y)))$ |

# Reasoning *within* Combined Logics: Epistemic & Spatial

A trivial problem for RCC:

| | |
|---|---|
| Catalunya is a border region of Spain | (*tpp* catalunya spain), |
| Spain and France share a border | (*ec* spain france), |
| Paris is a region inside France | (*ntpp* paris france) |
| | $\vdash^{\text{HOL}}$ |
| Catalunya and Paris are disconnected | (*dc* catalunya paris) |
| | $\wedge$ |
| Spain and Paris are disconnected | (*dc* spain paris) |

A trivial problem for RCC:

| | |
|---|---|
| Catalunya is a border region of Spain | $(tpp\ \text{catalunya spain})$, |
| Spain and France share a border | $(ec\ \text{spain france})$, |
| Paris is a region inside France | $(ntpp\ \text{paris france})$ |
| | $\vdash^{\text{HOL}}_{2.3s}$ |
| Catalunya and Paris are disconnected | $(dc\ \text{catalunya paris})$ |
| | $\wedge$ |
| Spain and Paris are disconnected | $(dc\ \text{spain paris})$ |

# Reasoning *within* Combined Logics: Epistemic & Spatial

$$\begin{array}{ll} & \text{valid } \forall\phi.\ \Box_{\text{fool}}\,\phi \supset \Box_{\text{bob}}\,\phi, \\ & \text{valid } \Box_{\text{fool}}\,(\lambda W.(ec \text{ spain france})), \\ & \text{valid } \Box_{\text{bob}}\,(\lambda W.(tpp \text{ catalunya spain})), \\ & \text{valid } \Box_{\text{bob}}\,(\lambda W.(ntpp \text{ paris france})) \\ \vdash^{HOL} & \text{valid } \Box_{\text{bob}}\,(\lambda W.((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris}))) \end{array}$$

# Reasoning *within* Combined Logics: Epistemic & Spatial

$$
\begin{array}{ll}
 & \text{valid } \forall\phi\text{.}\ \Box_{\text{fool}}\ \phi\ \supset\ \Box_{\text{bob}}\ \phi, \\
 & \text{valid } \Box_{\text{fool}}\ (\lambda W\text{.}(ec\ \text{spain france})), \\
 & \text{valid } \Box_{\text{bob}}\ (\lambda W\text{.}(tpp\ \text{catalunya spain})), \\
 & \text{valid } \Box_{\text{bob}}\ (\lambda W\text{.}(ntpp\ \text{paris france})) \\
\vdash^{HOL}_{20.4s} & \text{valid } \Box_{\text{bob}}\ (\lambda W\text{.}((dc\ \text{catalunya paris}) \wedge (dc\ \text{spain paris})))
\end{array}
$$

# Reasoning *within* Combined Logics: Epistemic & Spatial

$$\text{valid } \forall \phi_{\blacksquare}\ \Box_{\text{fool}}\, \phi \supset \Box_{\text{bob}}\, \phi,$$

$$\text{valid } \Box_{\text{fool}}\, (\lambda W_{\blacksquare} (ec \text{ spain france})),$$

$$\text{valid } \Box_{\text{bob}}\, (\lambda W_{\blacksquare} (tpp \text{ catalunya spain})),$$

$$\text{valid } \Box_{\text{bob}}\, (\lambda W_{\blacksquare} (ntpp \text{ paris france}))$$

$$\vdash^{HOL}_{20.4s} \text{valid } \Box_{\text{bob}}\, (\lambda W_{\blacksquare} ((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))$$

$$\not\vdash^{HOL} \text{valid } \Box_{\text{fool}}\, (\lambda W_{\blacksquare} ((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))$$

# Reasoning *within* Combined Logics: Epistemic & Spatial

$$
\begin{array}{ll}
 & \text{valid}\ \forall\phi.\ \Box_{\text{fool}}\,\phi \supset \Box_{\text{bob}}\,\phi, \\
 & \text{valid}\ \Box_{\text{fool}}\,(\lambda W.(ec\ \text{spain france})), \\
 & \text{valid}\ \Box_{\text{bob}}\,(\lambda W.(tpp\ \text{catalunya spain})), \\
 & \text{valid}\ \Box_{\text{bob}}\,(\lambda W.(ntpp\ \text{paris france})) \\
\vdash^{HOL}_{20.4s} & \text{valid}\ \Box_{\text{bob}}\,(\lambda W.((dc\ \text{catalunya paris}) \wedge (dc\ \text{spain paris}))) \\
\nvdash^{HOL}_{39.7s} & \text{valid}\ \Box_{\text{fool}}\,(\lambda W.((dc\ \text{catalunya paris}) \wedge (dc\ \text{spain paris})))
\end{array}
$$

# Reasoning *within* Combined Logics: Epistemic & Spatial

$$
\begin{array}{rl}
& \text{valid } \forall\phi\text{.}\ \Box_{\text{fool}}\,\phi \supset \Box_{\text{bob}}\,\phi, \\
& \text{valid } \Box_{\text{fool}}\,(\lambda W\text{.}(ec \text{ spain france})), \\
& \text{valid } \Box_{\text{bob}}\,(\lambda W\text{.}(tpp \text{ catalunya spain})), \\
& \text{valid } \Box_{\text{bob}}\,(\lambda W\text{.}(ntpp \text{ paris france})) \\
\vdash^{HOL}_{20.4s} & \text{valid } \Box_{\text{bob}}\,(\lambda W\text{.}((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris}))) \\
\nvdash^{HOL}_{39.7s} & \text{valid } \Box_{\text{fool}}\,(\lambda W\text{.}((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))
\end{array}
$$

Key idea is "Lifting" of RCC propositions to modal predicates:

$$
\underbrace{(tpp \text{ catalunya spain})}_{\text{type } o} \longrightarrow \underbrace{(\lambda W\text{.}(tpp \text{ catalunya spain}))}_{\text{type } \iota\to o}
$$

# LEO-II

(EPRSC grant EP/D070511/1 at Cambridge University)

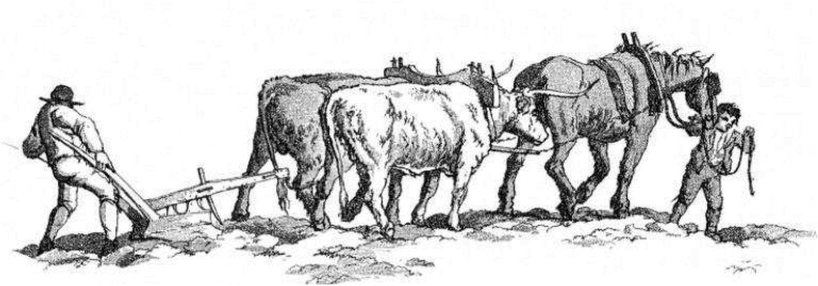
Thanks to Larry Paulson

# LEO-II

An Effective Higher-Order Theorem Prover

UNIVERSITY OF CAMBRIDGE

UNIVERSITÄT DES SAARLANDES



LEO-II employs FO-ATPs: E, Spass, Vampire

http://www.ags.uni-sb.de/~leo

# Conclusion

- HOL is suited as framework for combining logics
- automation of object-/meta-level reasoning — scalability ?
- embeddings can possibly be fully verified in Isabelle/HOL?
  (consistency of QML embedding: 3.8s – IsabelleN)
- current work: application to ontology reasoning (SUMO)

You can use this framework right away! Try it!

- new TPTP infrastructure for automated HOL reasoning
  (jww Geoff Sutcliffe / EU grant THFTPTP)
  - standardized input / output language (THF)
  - problem library: 3000 problems
  - yearly CASC competitions
- provers and examples are online; demo: `http://tptp.org`
  Wise Men Puzzle:

`http://www.cs.miami.edu/~tptp/cgi-bin/SeeTPTP?Category=Problems&Domain=PUZ&File=PUZ087^1.p`

| Problem | LEO-II | TPS | IsaP |
|---|---|---|---|
| Reasoning about Logics and Combined Logics | | | |
| 1 | 0.0 | 0.3 | 3.6 |
| 2 | 0.0 | 0.3 | 13.9 |
| 3 | 0.0 | 0.3 | 4.0 |
| 4 | 0.0 | 0.3 | 15.9 |
| 5 | 0.1 | 0.3 | 16.0 |
| 6 | 0.0 | 0.3 | 3.6 |
| 7 | 0.1 | 51.2 | 3.9 |
| 8 | 0.1 | 0.3 | 3.9 |
| 9 | 0.1 | 0.3 | 3.6 |
| 10 | 0.1 | 0.3 | 4.1 |
| 11 | 0.0 | 0.3 | 3.7 |
| 12 | — | 0.3 | 53.8 |
| 13 | 0.0 | 0.3 | 3.7 |
| 14 | 0.0 | 0.3 | 3.8 |
| 15 | — | 0.8 | 67.0 |
| 16 | 1.6 | 0.3 | 29.3 |
| 17 | 37.9 | — | — |
| 18 | — | 6.6 | — |
| 19 | — | — | — |
| 20 | 0.1 | 0.4 | 8.1 |
| 21 | 0.1 | 0.4 | 4.3 |
| 22 | 0.2 | 27.4 | 4.0 |
| 23 | 0.1 | 8.9 | 4.0 |
| 24 | 0.1 | 1.2 | 3.7 |
| 26 | 0.1 | 1.7 | 4.2 |
| 27 | 0.2 | 14.8 | 5.4 |
| 28 | 0.1 | 0.6 | 3.7 |

| Problem | LEO-II | TPS | IsaP |
|---|---|---|---|
| Reasoning about Logics and Combined Logics | | | |
| 29 | 0.2 | 2.3 | 4.0 |
| 30 | 0.1 | 0.9 | 3.9 |
| 31 | 0.1 | 12.8 | 16.5 |
| 32 | Countersatisfiable — | — | — |
| 33 | 0.0 | 0.3 | 3.6 |
| 34 | 0.0 | 0.3 | 3.6 |
| 35 | 0.1 | 0.4 | 3.6 |
| 36 | 0.2 | 35.5 | — |
| 37 | 0.4 | — | — |
| Reasoning within Combined Logics | | | |
| 38 | 0.1 | — | 102.4 |
| 39 | 0.3 | — | — |
| 40 | 2.3 | — | 112.7 |
| 42 | 20.4 | — | — |
| 42 | Countersatisfiable — | — | — |