

# Higher-Order Modal Logics (Lectures 1&2)

Christoph Benz Müller<sup>1</sup> and Bruno Woltzenlogel-Paleo

```
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p

Leo-II tries to prove
=====

Goedel's Theorem T3: "Necessarily, God exists"
thf(thmT3,conjecture,
  ( v
    @ ( mbox
      @ ( mexists_ind
        @ ^ [X: mu] :
          ( g @ X ) ) ) ).

Assumptions: D1, C, T2, D3, A5

. searching for proof ..

*****
*   Proof found   *
*****
% SZS status Theorem for Notwendigerweise-existiert-Gott.p

. generating proof object □
```

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<sup>1</sup>Supported by DFG Heisenberg Fellowship BE 2501/9-1/2



## Introduction

This Tutorial ...

... is about (narrow picture)

- ▶ higher-order modal logic (HOML)
- ▶ classical higher-order logic (HOL)
- ▶ embedding of HOML in HOL
- ▶ mechanisation and automation with HOL ATPs
- ▶ various applications, including metaphysics

... is about (wider picture)

- ▶ a quite universal approach to automate reasoning for a wide range of classical and non-classical logics
- ▶ a very broad range of possible applications
- ▶ including meta-reasoning about logical systems

Our primary interest . . .

- ▶ is in expressive logics:  
quantification (first- and higher-order) and lambda-expressions
- ▶ propositional fragments are trivially covered



## Introduction: Outline of Tutorial

### Lecture 1 & 2

- ▶ Motivation, Logic Embedding Approach – Advantages
- ▶ Introduction to HOL
  - ▶ Syntax, Henkin and Standard Semantics, Calculi (and more)
  - ▶ Automated Theorem Provers for HOL, TPTP THF Infrastructure

### Lecture 3

- ▶ Introduction to HOML
  - ▶ Syntax, Henkin and Standard Semantics
  - ▶ Embedding in HOL
  - ▶ Automation with HOL-ATPs
  - ▶ Flexibility

### Lecture 4

- ▶ Automating the Ontological Argument for the Existence of God

### Lecture 5

- ▶ Interacting with Modal Logics in the Coq Proof Assistant
- ▶ ?(Embedding Quantified Conditional Logics)
- ▶ ?(Cut-Elimination and Cut-Simulation in HOL and HOML)
- ▶ ?(SUMO Ontology and HOML)

## Introduction: Expressivity Matters — Cantor's Theorem

**Cantor's theorem:** The set of all subsets of A, that is, the power set of A, has a strictly greater cardinality than A itself.

In HOL Cantor's theorem (surjective version) can be encoded as

$$\neg \exists f_{t \rightarrow (t \rightarrow o)} \forall g_{t \rightarrow o} \exists x_t. f x = g$$

HO ATPs can solve this problem very efficiently.

Their solution includes the detection and application of the diagonalisation argument.

Today: basic test example for new higher-order theorem provers.

Further reading: [AndrewsEtAl., Automating higher-order logic, 1984]

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## Introduction: Expressivity Matters — Boolos' Example

[George Boolos, A curious inference, J. Philosophical Logic, 16:1-12, 1987]

1.  $\forall n \, f(n, 1) = s(1)$
2.  $\forall x \, f(1, s(x)) = s(s(f(1, x)))$
3.  $\forall n \, \forall x \, f(s(n), s(x)) = f(n, f(s(n), x))$
4.  $D(1)$
5.  $\forall x \, D(x) \rightarrow D(s(x))$
- $\therefore$
6.  $D(f(s(s(s(s(1))))), s(s(s(s(1))))))$

Induction proof: from (4) and (5), we get  $\forall x \, D(x)$ , hence  $D(f(s(s(s(s(1))))), s(s(s(s(1))))))$  by  $\forall$ -elimination.

But induction is not given, hence the first order proof consists of brute force modus ponens applications: infeasible number of single steps ( $2^{(2^{\dots 2})}$  with 64K '2s')

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### Comprehension axioms

$$\exists N_{\alpha^n \rightarrow \beta} \forall \overline{z^n} N(z^1, \dots, z^n) = B_\beta$$

Can be avoided: use  $\lambda$ -binding construct to denote  $N$

Instances of comprehension axioms in Boolos' proof:

$$\exists N \forall z N(z) = (\forall X X(1) \wedge \forall y (X(y) \rightarrow X(s(y))) \rightarrow X(z))$$

$$\exists E \forall z E(z) = (N(z) \wedge D(z))$$

**Central idea:** “assume the induction principle holds for number  $z$  – corresponding to  $N(z)$  – then we can show for any predicate  $X$  a property  $X(z)$  by induction.”

The proof employs the following lemmata:

**Lemma 1:**  $N(1), \forall y (N(y) \rightarrow N(s(y))), N(s(s(s(1))))), E(1), \forall y (E(y) \rightarrow E(s(y))), E(s(1))$

**Lemma 2:**  $\forall n N(n) \rightarrow \forall x (N(x) \rightarrow E(f(n, x)))$

The theorem itself is then an easy application of the two lemmata.

# Introduction: Expressivity Matters — Boolos' Example

<sup>1</sup>By the comprehension principle of second order logic,  $\exists N \forall z (Nz \leftrightarrow \forall x [X1 \& \forall y (Xy \rightarrow Xsy) \rightarrow Xz])$ , <sup>2</sup>and then for some  $N$ ,  $\exists E \forall z (Ez \leftrightarrow Nz \& Dz)$ .  
<sup>3</sup>**Lemma 1:** <sup>3.1</sup> $N1$ ; <sup>3.2</sup> $\forall y (Ny \rightarrow Nsy)$ ; <sup>3.3</sup> $Nsssl$ ; <sup>3.4</sup> $E1$ ; <sup>3.5</sup> $\forall y (Ey \rightarrow Esy)$ ; <sup>3.6</sup> $Es1$ ;  
<sup>4</sup>**Lemma 2:**  $\forall n (Nn \rightarrow (\forall x (Nx \rightarrow Efnx)))$   
*Proof:* <sup>4.1</sup>By comprehension,  $\exists M \forall n (Mn \leftrightarrow \forall x (Nx \Rightarrow Efnx))$ . <sup>4.2</sup>We want  $\forall n (Nn \rightarrow Mn)$ . <sup>4.3</sup>Enough to show <sup>4.3.1</sup> $M1$  and <sup>4.3.2</sup> $\forall n (Mn \rightarrow Msn)$ , for then if <sup>4.4</sup> $Nn$ , <sup>4.5</sup> $Mn$ .  
<sup>4.3.1</sup> $M1$ : <sup>4.3.1.1</sup>Want  $\forall x (Nx \rightarrow Efx)$ . <sup>4.3.1.2</sup>By comprehension,  $\exists Q \forall x (Qx \leftrightarrow Efx)$ .  
<sup>4.3.1.3</sup>Want  $\forall x (Nx \rightarrow Qx)$ . <sup>4.3.1.4</sup>Enough to show <sup>4.3.1.4.1</sup> $Q1$  and <sup>4.3.1.4.2</sup> $\forall x (Qx \rightarrow Qsx)$ .  
<sup>4.3.1.4.1</sup> $Q1$ : <sup>4.3.1.4.1.1</sup>Want  $Ef1$ . <sup>4.3.1.4.1.2</sup>But  $f11 = s1$  by (1) and <sup>4.3.1.4.1.3</sup> $Es1$  by Lemma 1.  
<sup>4.3.1.4.2</sup> $\forall x (Qx \rightarrow Qsx)$ : <sup>4.3.1.4.2.1</sup>Suppose  $Qx$ , <sup>4.3.1.4.2.2</sup>i.e.  $Efx$ . <sup>4.3.1.4.2.3</sup>By (2)  $f1sx = sfx$ ; <sup>4.3.1.4.2.4</sup>by Lemma 1 twice,  $Efx$ . <sup>4.3.1.4.2.5</sup>Thus  $Qsx$  and <sup>4.3.1.4.2.6</sup> $M1$ .  
<sup>4.3.2</sup> $\forall n (Mn \rightarrow Msn)$ : <sup>4.3.2.1</sup>Suppose  $Mn$ , <sup>4.3.2.2</sup>i.e.  $\forall x (Nx \rightarrow Efnx)$ .  
<sup>4.3.2.3</sup>Want  $Msn$ , <sup>4.3.2.4</sup>i.e.  $\forall x (Nx \rightarrow Efsnx)$ . <sup>4.3.2.5</sup>By comprehension,  $\exists P \forall x (Px \leftrightarrow Efsnx)$ .  
<sup>4.3.2.6</sup>Want  $\forall x (Nx \rightarrow Px)$ . <sup>4.3.2.7</sup>Enough to show <sup>4.3.2.7.1</sup> $P1$  and <sup>4.3.2.7.2</sup> $\forall x (Px \rightarrow Psx)$ .  
<sup>4.3.2.7.1</sup> $P1$ : <sup>4.3.2.7.1.1</sup>Want  $Efsn1$ . <sup>4.3.2.7.1.2</sup>But  $fsn1 = s1$  by (1) and <sup>4.3.2.7.1.3</sup> $Es1$  by Lemma 1.  
<sup>4.3.2.7.2</sup> $\forall x (Px \rightarrow Psx)$ : <sup>4.3.2.7.2.1</sup>Suppose  $Px$ , <sup>4.3.2.7.2.2</sup>i.e.  $Efsnx$ ; <sup>4.3.2.7.2.3</sup>thus  $Nfsnx$ . <sup>4.3.2.7.2.4</sup>Want  $Efsnsx$ . <sup>4.3.2.7.2.5</sup>Since  $Nfsnx$  and  $Mn$ ,  $Efnfsnx$ .  
<sup>4.3.2.7.2.6</sup>But by (3)  $fnfsnx = fnsnx$ ; <sup>4.3.2.7.2.7</sup>thus  $Efsnsx$ .  
<sup>5</sup>By Lemma 1,  $Nsssl$ . <sup>6</sup>By Lemma 2,  $Efssslsssl$ . <sup>7</sup>Thus,  $Dfssslsssl$ , as desired.

Formalization in OMEGA and Mizar: see [BenzmüllerBrown, The curious inference of Boolos in MIZAR and OMEGA, Studies in Logic, Grammar, and Rhetoric, volume 10(23), pp. 299-388, 2007.]

Earlier paper: [BenzmüllerKerber, A Lost Proof, 2001].

Earlier poster: <http://christoph-benzmueller.de/papers/poster-tphols01.pdf>

## Introduction: Higher-Order Modal Logics

$\Box P$ :  $P$  is necessary,  $P$  is obligatory,  $P$  is known,  $P$  is believed, always  $P$  ...

$\Diamond P$ :  $P$  is possible,  $P$  is permissible,  $P$  is epistemically possible,  $P$  is doxastically possible, eventually  $P$  ...

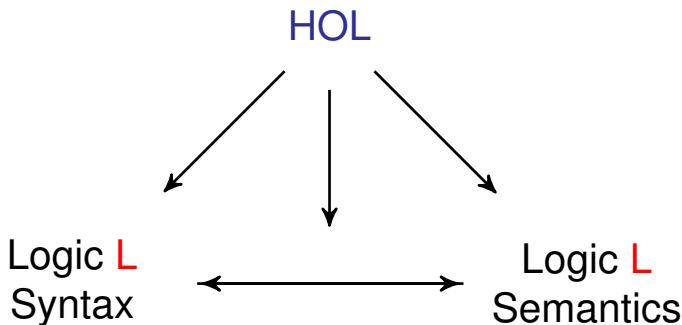
$\Box$  and  $\Diamond$  are not truth-functional

HOL can be extended by  $\Box$  and  $\Diamond$  to obtain HOML

There are many interesting applications of such logics, e.g. in Metaphysics



## Introduction: HOL as a Universal (Meta-)Logic via Semantic Embeddings

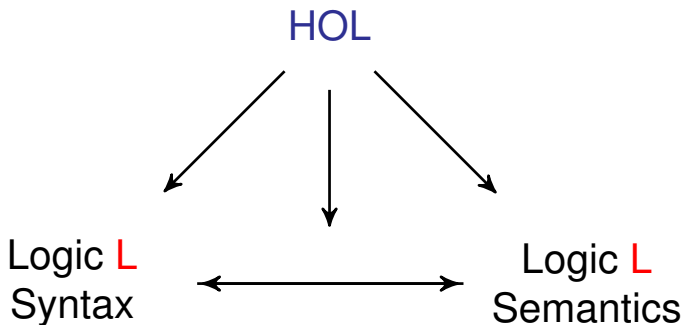


Examples for **L** we have already studied:

Modal Logics, Conditional Logics, Intuitionistic Logics, Access Control Logics, Nominal Logics, Multivalued Logics (SIXTEEN), Logics based on Neighborhood Semantics, (Mathematical) Fuzzy Logics, Paraconsistent Logics, ...

Works also for (first-order & higher-order) quantifiers

## Introduction: HOL as a Universal (Meta-)Logic via Semantic Embeddings



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**Works also for (first-order & higher-order) quantifiers**

## Introduction: Embedding Approach — Idea

HOL (meta-logic)

$\varphi ::=$  

Your-logic (object-logic)

$\psi ::=$  

Embedding of  in 

 = 

 = 

 = 

 = 

Embedding of meta-logical notions on  in 

*valid* = 

*satisfiable* = 

*...* = 

Pass this set of equations to a higher-order automated theorem prover

## Introduction: Embedding Approach — HOML in HOL

**HOML**       $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma \varphi \mid \exists x_\gamma \varphi$

**HOL**       $s, t ::= C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha t_o$

**HOML in HOL:** **HOML** formulas  $\varphi$  are mapped to **HOL** predicates  $\varphi_{t \rightarrow o}$   
(explicit representation of labelled formulas)

|               |     |                                                                                                           |
|---------------|-----|-----------------------------------------------------------------------------------------------------------|
| $\neg$        | $=$ | $\lambda\varphi_{t \rightarrow o} \lambda w_t \neg \varphi w$                                             |
| $\wedge$      | $=$ | $\lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda w_t (\varphi w \wedge \psi w)$    |
| $\rightarrow$ | $=$ | $\lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda w_t (\neg \varphi w \vee \psi w)$ |
| $\forall$     | $=$ | $\lambda h_{\gamma \rightarrow (t \rightarrow o)} \lambda w_t \forall d_\gamma h d w$                     |
| $\exists$     | $=$ | $\lambda h_{\gamma \rightarrow (t \rightarrow o)} \lambda w_t \exists d_\gamma h d w$                     |
| $\Box$        | $=$ | $\lambda\varphi_{t \rightarrow o} \lambda w_t \forall u_t (\neg R w u \vee \varphi u)$                    |
| $\Diamond$    | $=$ | $\lambda\varphi_{t \rightarrow o} \lambda w_t \exists u_t (R w u \wedge \varphi u)$                       |
| <b>valid</b>  | $=$ | $\lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$                                                  |

**Ax** (polymorphic over  $\gamma$ )

The equations in **Ax** are given as axioms to the **HOL** provers!

# Advantages of the Embedding Approach

1. **Pragmatics and convenience:**
  - implementing new provers made simple (even for not yet automated logics)
2. **Availability:**
  - simply reuse and adapt our existing encodings (THF, Isabelle/HOL, Coq)
3. **Flexibility:**
  - rapid experimentation with logic variations and logic combinations
4. **Relation to labelled deductive systems:**
  - extra-logical labels vs. intra-logical labels (here)
5. **Relation to standard translation:**
  - extra-logical translation vs. extended intra-logical translation (here)
6. **Meta-logical reasoning:**
  - various examples already exist, e.g. verification of modal logic cube
7. **Direct calculi and user intuition:**
  - possible: tactics on top of embedding, hiding of embedding
8. **Soundness and completeness:**
  - already proven for many non-classical logics (wrt Henkin semantics)
9. **Cut-elimination:**
  - generic indirect result, since HOL enjoys cut-elimination (Henkin semantics)

## Advantage: 1. Pragmatics and convenience

implementing new provers made simple (even for not yet automated logics)

### A very “Lean” Prover for HOML KB

```
1  %----The base type $i (already built-in) stands here for worlds and
2  %----mu for individuals; $o (also built-in) is the type of Booleans
3  thf(mu_type,type,(mu:$tType)).
4  %----Reserved constant r for accessibility relation
5  thf(r,type,(r:$i>$i>$o)).
6  %----Modal logic operators not, or, and, implies, box, diamond
7  thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
8  thf(mnot,definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))).
9  thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
10 thf(mor,definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)|(Psi@W))))).
11 thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
12 thf(mand,definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
13 thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
14 thf(mimplies,definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
15 thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
16 thf(mbox,definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V)|(A@V))))).
17 thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
18 thf(mdia,definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V)&(A@V))))).
19 %----Quantifiers (constant domains) for individuals and propositions
20 thf(mforall_ind_type,type,(mforall_ind:(mu>$i>$o)>$i>$o)).
21 thf(mforall_ind,definition,(mforall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
22 thf(mforall_indset_type,type,(mforall_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
23 thf(mforall_indset,definition,(mforall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(A@X@W)))).
24 thf(mexists_ind_type,type,(mexists_ind:(mu>$i>$o)>$i>$o)).
25 thf(mexists_ind,definition,(mexists_ind = (^[A:mu>$i>$o,W:$i]:?[X:mu]:(A@X@W)))).
26 thf(mexists_indset_type,type,(mexists_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
27 thf(mexists_indset,definition,(mexists_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:?[X:mu>$i>$o]:(A@X@W)))).
28 %----Definition of validity (grounding of lifted modal formulas)
29 thf(v_type,type,(v:($i>$o)>$o)).
30 thf(mvalid,definition,(v = (^[A:$i>$o]:![W:$i]:(A@W)))).
31 %----Properties of accessibility relations: symmetry
32 thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
33 thf(msymmetric,definition,(msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T)=>(R@T@S))))).
34 %----Here we work with logic KB, i.e., we postulate symmetry for r
35 thf(sym,axiom,(msymmetric@r)).
```

TPTP THF0 syntax:

[SutcliffeBenzmüller, J.Formalized Reasoning, 2010]

## Advantage: 1. Pragmatics and convenience

implementing new provers made simple (even for not yet automated logics)

### Approach is competitive

- ▶ First-order modal logic: see experiments in

[BenzmüllerOttenRaths, ECAI, 2012]

[BenzmüllerRaths, LPAR, 2013]

[Benzmüller, ARQNL, 2014]

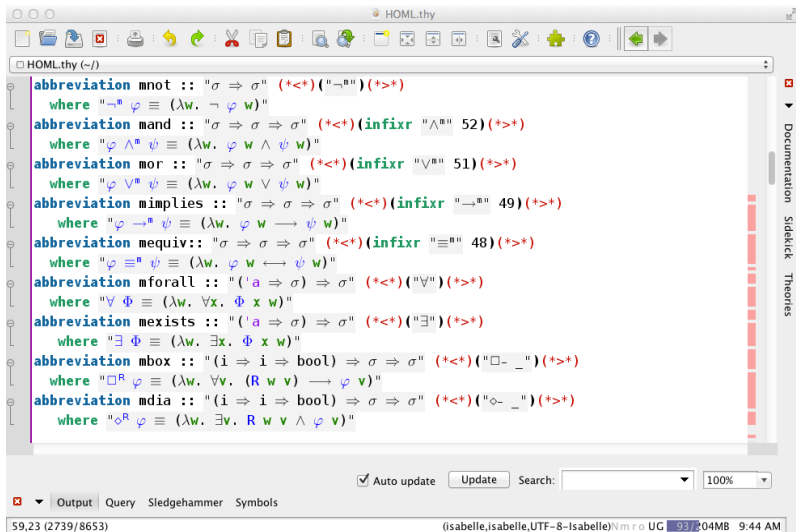
- ▶ Higher-order modal logics:

There are no other systems yet!

## Advantage: 2. Availability

simply reuse and adapt our existing encodings (THF, Isabelle/HOL, Coq)

## HOML in Isabelle/HOL



```
abbreviation mnot :: " $\sigma \Rightarrow \sigma$ " (*<*) ("¬") (*>*)
  where "¬  $\varphi \equiv (\lambda w. \neg \varphi w)$ "
abbreviation mand :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " (*<*) (infixr "∧" 52) (*>*)
  where " $\varphi \wedge \psi \equiv (\lambda w. \varphi w \wedge \psi w)$ "
abbreviation mor :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " (*<*) (infixr "∨" 51) (*>*)
  where " $\varphi \vee \psi \equiv (\lambda w. \varphi w \vee \psi w)$ "
abbreviation mimplies :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " (*<*) (infixr "→" 49) (*>*)
  where " $\varphi \rightarrow \psi \equiv (\lambda w. \varphi w \rightarrow \psi w)$ "
abbreviation mequiv :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " (*<*) (infixr "≡" 48) (*>*)
  where " $\varphi \equiv \psi \equiv (\lambda w. \varphi w \leftrightarrow \psi w)$ "
abbreviation mforall :: " $(\lambda a. \sigma) \Rightarrow \sigma$ " (*<*) ("∀") (*>*)
  where " $\forall \Phi \equiv (\lambda w. \forall x. \Phi x w)$ "
abbreviation mexists :: " $(\lambda a. \sigma) \Rightarrow \sigma$ " (*<*) ("∃") (*>*)
  where " $\exists \Phi \equiv (\lambda w. \exists x. \Phi x w)$ "
abbreviation mbox :: " $(i \Rightarrow i \Rightarrow \text{bool}) \Rightarrow \sigma \Rightarrow \sigma$ " (*<*) ("□" _ ) (*>*)
  where " $\Box^R \varphi \equiv (\lambda w. \forall v. (R w v) \rightarrow \varphi v)$ "
abbreviation mdia :: " $(i \Rightarrow i \Rightarrow \text{bool}) \Rightarrow \sigma \Rightarrow \sigma$ " (*<*) ("◇" _ ) (*>*)
  where " $\Diamond^R \varphi \equiv (\lambda w. \exists v. R w v \wedge \varphi v)$ "
```

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### Advantage: 3. Flexibility

rapid experimentation with logic variations and logic combinations

## Postulating modal axioms or semantical constraints

HOL

### Sahlqvist axioms

- M: valid  $\forall \varphi (\Box^R \varphi \rightarrow \varphi)$   
B: valid  $\forall \varphi (\varphi \rightarrow \Box^R \Diamond^R \varphi)$   
D: valid  $\forall \varphi (\Box^R \varphi \rightarrow \Diamond^R \varphi)$   
4: valid  $\forall \varphi (\Box^R \varphi \rightarrow \Box^R \Box^R \varphi)$   
5: valid  $\forall \varphi (\Diamond^R \varphi \rightarrow \Box^R \Diamond^R \varphi)$

$\leftrightarrow$

$\leftrightarrow$

$\leftrightarrow$

$\leftrightarrow$

$\leftrightarrow$

### Semantical constraints

- $\forall x (Rxx)$  (reflexivity)  
 $\forall x \forall y (Rxy \rightarrow Ryx)$  (symmetry)  
 $\forall x \exists y (Rxy)$  (serial)  
 $\forall x \forall y \forall z (Rxy \wedge Ryz \rightarrow Rxz)$  (transitivity)  
 $\forall x \forall y \forall z (Rxy \wedge Rxz \rightarrow Ryz)$  (euclidean)

### Advantage: 3. Flexibility

rapid experimentation with logic variations and logic combinations

## Possibilist vs. Actualist Quantification

$$\forall = \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_{\iota} \forall d_{\gamma} h d w \quad (\text{constant domains})$$

becomes

$$\forall^{va} = \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_{\iota} \forall d_{\gamma} (\mathbf{ExInW} d w \rightarrow h d w) \quad (\text{varying domains})$$

where **ExInW** is an existence predicate

(additional axioms: non-empty domains, denotation of constants & functions)

#### Advantage: 4. Relation to labelled deductive systems



#### Advantage: 4. Relation to labelled deductive systems



$$\Diamond \exists x G(x) \text{ worldlabel} \longrightarrow ((\Diamond \exists x G(x))_{t \rightarrow o} \text{worldlabel}_\mu)$$

## Advantage: 5. Relation to standard translation

extra-logical translation vs. extended intra-logical translation (here)

[BenzmüllerPaulson, LogicaUniversalis, 2013]

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

### Intra-logical realisation of the standard translation

$(\Box\varphi)$  **a**

$\longrightarrow ((\Box\varphi)_{l \rightarrow o} a)$

$\longrightarrow (((\lambda\varphi_{l \rightarrow o} \lambda w_l \forall u_l (\neg R w u \vee \varphi u)) \varphi)_{l \rightarrow o} a)$

$\longrightarrow (\forall u_l (\neg R a u \vee \varphi_{l \rightarrow o} u))$

We have extended this also for first-order and higher-order quantifiers!

$(\forall x \varphi(x))$  **a**

$\longrightarrow ((\forall x \varphi(x))_{l \rightarrow o} a)$

$\longrightarrow ((\forall (\lambda x \varphi(x)))_{l \rightarrow o} a)$

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## Advantage: 5. Relation to standard translation

extra-logical translation vs. extended intra-logical translation (here)

[BenzmüllerPaulson, LogicaUniversalis, 2013]

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

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We have extended this also for first-order and higher-order quantifiers!

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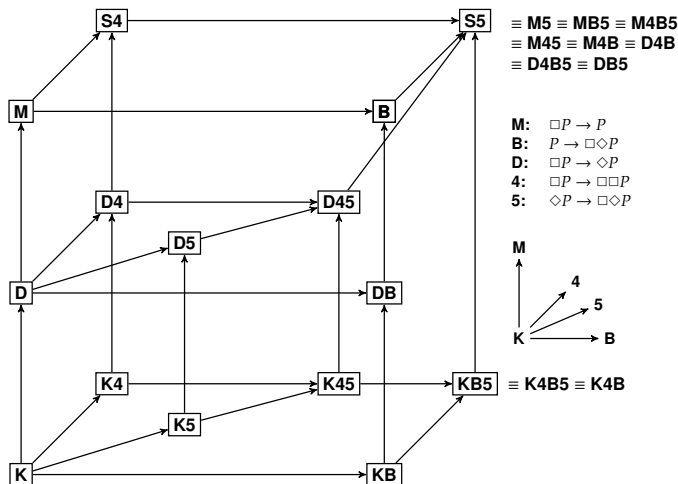
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## Advantage: 6. Meta-logical reasoning

various examples already exist, e.g. verification of modal logic cube

[Benzmüller, FestschriftWalther, 2010]

[BenzmüllerClausSultana, PxTP, 2015]



Verification of cube in less than 1 minute in Isabelle/HOL

## Advantage: 7. Direct calculi and user intuition

abstract level tactics (here in Coq) on top of embedding, hiding of embedding

[BenzmüllerWoltzenlogelPaleo, CSR'2015]

**Lemma** mp\_dia:

[mforall p, mforall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].

**Proof.** mv.

intros p q H1 H2. dia\_e H1. dia\_i w0. box\_e H2 H3. apply H3. exact H1.

**Qed.**

$$\begin{array}{c}
 \overline{\Diamond p} \quad 1 \quad \overline{\Box(p \rightarrow q)} \quad 2 \\
 \overline{\Diamond p} \quad \Diamond_E \quad \overline{\Box(p \rightarrow q)} \quad \Box_E \\
 \\
 w_0 \quad \boxed{\begin{array}{c} p \quad p \rightarrow q \\ \hline q \end{array}} \rightarrow_E \\
 \\
 \overline{\Diamond q} \quad \Diamond_I \\
 \hline \overline{\Diamond p \rightarrow (\Box(p \rightarrow q)) \rightarrow (\Diamond q)} \rightarrow_I^1, \rightarrow_I^2 \\
 \hline \overline{\forall p. \forall q. \Diamond p \rightarrow (\Box(p \rightarrow q)) \rightarrow \Diamond q} \forall_I, \forall_I
 \end{array}$$

## Advantage: 8. Soundness and completeness

already proven for many non-classical logics (wrt Henkin semantics)

### Soundness and Completeness

$$\models^L \varphi \quad \text{iff} \quad \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } \varphi_{l \rightarrow o}$$

Logic L:

- ▶ Higher-order Modal Logics [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]
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- ▶ ... more is on the way ... including:
  - ▶ Description Logics
  - ▶ Nominal Logics
  - ▶ Multivalued Logics (SIXTEEN)
  - ▶ Logics based on Neighborhood Semantics
  - ▶ (Mathematical) Fuzzy Logics
  - ▶ Paraconsistent Logics



## Advantage: 9. Cut-elimination

generic indirect result, since HOL enjoys cut-elimination (Henkin semantics)

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### Soundness and Completeness and Cut-elimination

$$\models^L \varphi \quad \text{iff} \quad \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } \varphi_{l \rightarrow o} \quad \text{iff} \quad \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{l \rightarrow o}$$

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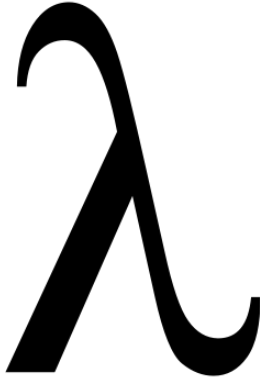
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One-sided sequent calculus  $\mathcal{G}_{\beta\text{fb}}$  [BenzmüllerBrownKohlhase, LMCS, 2009]

( $\Delta$ : finite sets of  $\beta$ -normal closed formulas,  $\Delta * \mathbf{A}$  stands for  $\Delta \cup \{\mathbf{A}\}$ ,

$\text{cwff}_\alpha$ : set of closed terms of type  $\alpha$ ,  $\doteq$  abbreviates Leibniz equality):

$$\begin{array}{l}
 \text{Base Rules} \quad \frac{\mathbf{A} \text{ atomic (\& } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg \mathbf{A}} \mathcal{G}(\text{init}) \quad \frac{\Delta * \mathbf{A}}{\Delta * \neg \neg \mathbf{A}} \mathcal{G}(\neg) \quad \frac{\Delta * \neg \mathbf{A} \quad \Delta * \neg \mathbf{B}}{\Delta * \neg (\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_-) \\
 \\
 \frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_+) \quad \frac{\Delta * \neg (\mathbf{A} \mathbf{C}) \downarrow_\beta \quad \mathbf{C} \in \text{cwff}_\alpha}{\Delta * \neg \Pi^\alpha \mathbf{A}} \mathcal{G}(\Pi_-^c) \quad \frac{\Delta * (\mathbf{A} \mathbf{c}) \downarrow_\beta \quad c_{\alpha \text{ new}}}{\Delta * \Pi^\alpha \mathbf{A}} \mathcal{G}(\Pi_+^c) \\
 \\
 \text{Full Extensionality} \quad \frac{\Delta * (\forall X_\alpha. \mathbf{A} X \doteq^\beta \mathbf{B} X) \downarrow_\beta}{\Delta * (\mathbf{A} \doteq^{\alpha \rightarrow \beta} \mathbf{B})} \mathcal{G}(\text{f}) \quad \frac{\Delta * \neg \mathbf{A} * \mathbf{B} \quad \Delta * \neg \mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^\circ \mathbf{B})} \mathcal{G}(\text{b}) \\
 \\
 \text{Initial. and Decomp. of Leibniz Equality} \quad \frac{\Delta * (\mathbf{A} \doteq^\circ \mathbf{B}) \quad \mathbf{A}, \mathbf{B} \text{ atomic}}{\Delta * \neg \mathbf{A} * \mathbf{B}} \mathcal{G}(\text{Init}^\doteq) \\
 \\
 \frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \dots \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad n \geq 1, \beta \in \{o, \iota\}, h_{\alpha^n \rightarrow \beta} \in \Sigma}{\Delta * (h\overline{\mathbf{A}}^n \doteq^\beta h\overline{\mathbf{B}}^n)} \mathcal{G}(d)
 \end{array}$$



## Higher-order Logic (HOL)

# Classical Higher-Order Logic (HOL)

| Expressivity           | FOL | HOL | Example                                          |
|------------------------|-----|-----|--------------------------------------------------|
| Quantification over    |     |     |                                                  |
| - Individuals          | ✓   | ✓   | $\forall x p(f(x))$                              |
| - Functions            | ✗   | ✓   | $\forall f p(f(a))$                              |
| - Predicates/Sets/Rels | ✗   | ✓   | $\forall p p(f(a))$                              |
| Unnamed                |     |     |                                                  |
| - Functions            | ✗   | ✓   | $(\lambda x x)$                                  |
| - Predicates/Sets/Rels | ✗   | ✓   | $(\lambda x x \neq a)$                           |
| Statements about       |     |     |                                                  |
| - Functions            | ✗   | ✓   | <i>continuous</i> $(\lambda x x)$                |
| - Predicates/Sets/Rels | ✗   | ✓   | <i>reflexive</i> $(=)$                           |
| Powerful abbreviations | ✗   | ✓   | <i>reflexive</i> $= \lambda r \forall x r(x, x)$ |

# Classical Higher-Order Logic (HOL)

| Expressivity           | FOL | HOL | Example                                                                                                                            |
|------------------------|-----|-----|------------------------------------------------------------------------------------------------------------------------------------|
| Quantification over    |     |     |                                                                                                                                    |
| - Individuals          | ✓   | ✓   | $\forall x_l p_{l \rightarrow o}(f_{l \rightarrow l}(x_l))$                                                                        |
| - Functions            | ✗   | ✓   | $\forall f_{l \rightarrow l} p_{l \rightarrow o}(f_{l \rightarrow o}(a_l))$                                                        |
| - Predicates/Sets/Rels | ✗   | ✓   | $\forall p_{l \rightarrow o} p_{l \rightarrow o}(f_{l \rightarrow l}(a_l))$                                                        |
| Unnamed                |     |     |                                                                                                                                    |
| - Functions            | ✗   | ✓   | $(\lambda x_l x_l)$                                                                                                                |
| - Predicates/Sets/Rels | ✗   | ✓   | $(\lambda x_{l \rightarrow l} x_{l \rightarrow l} \neq_{l \rightarrow l \rightarrow p} a)_l)$                                      |
| Statements about       |     |     |                                                                                                                                    |
| - Functions            | ✗   | ✓   | $continuous_{(l \rightarrow l) \rightarrow o}(\lambda x_l x_l)$                                                                    |
| - Predicates/Sets/Rels | ✗   | ✓   | $reflexive_{(l \rightarrow l \rightarrow o) \rightarrow o}(=_{l \rightarrow l \rightarrow o})$                                     |
| Powerful abbreviations | ✗   | ✓   | $reflexive_{(l \rightarrow l \rightarrow o) \rightarrow o} =$<br>$\lambda r_{(l \rightarrow l \rightarrow o)} \forall x_l r(x, x)$ |

**Simple Types:** Prevent Some Paradoxes and Inconsistencies

## HOL: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

(we may add further base types, e.g.  $\mu$ )

HOL Language:

$$s, t ::= P_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ (\neg_{o \rightarrow o} s_o) \mid ((\forall_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha s_o)$$

constant symbols

variable symbols

lambda abstraction

application

negation

disjunction

universal quantification

Terms of type  $o$ : formulas

Other logical connectives can be defined, e.g.  $\exists x s$  stands for  $\neg \forall x \neg s$

Equality may also be defined:  $s \doteq t$  stands for  $\forall P (Ps \Rightarrow Pt)$   
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Terms of type  $o$ : formulas

Other logical connectives can be defined, e.g.  $\exists x s$  stands for  $\neg \forall x \neg s$

Equality may also be defined:  $s \doteq t$  stands for  $\forall P(Ps \Rightarrow Pt)$   
(but it is strongly recommended to add primitive equality!)

## HOL: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

(we may add further base types, e.g.  $\mu$ )

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$$s, t ::= P_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid$$

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(we may use syntactical sugar:  $\forall x_\alpha s$ )

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For each  $s_\alpha \in \mathcal{HOL}$  there is a unique  $\beta$ -normal form and a unique  $\beta\eta$ -normal form.

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## Semantics of HOL is (meanwhile) well understood

- ▶ Origin [Church, J.Symb.Log., 1940]
- ▶ Henkin-Semantics [Henkin, J.Symb.Log., 1950]  
[Andrews, J.Symb.Log., 1971, 1972]
- ▶ Extensionality/Intensionality [BenzmüllerEtAl., J.Symb.Log., 2004]  
[Muskens, J.Symb.Log., 2007]

HOL with Henkin-Semantics:

semi-decidable & compact & existence of countable models (like FOL)

Recommended Reading: [BenzmüllerMiller, HandbookHistoryOfLogic, Vol.9, 2014]

### A Frame $D$

is a collection  $\{D_\alpha\}_{\alpha \in T}$  of nonempty sets  $D_\alpha$ , such that

- ▶  $D_t$  can be chosen freely
- ▶  $D_o = \{T, F\}$  (for truth and falsehood), and
- ▶  $D_{\alpha \rightarrow \beta}$  are collections of functions mapping  $D_\alpha$  into  $D_\beta$

### A Model $M$

for HOL is a tuple  $M = \langle D, I \rangle$ , where

- ▶  $D$  is a frame;
- ▶  $I$  is a family of typed interpretation functions mapping constant symbols  $p_\alpha$  to appropriate elements of  $D_\alpha$ , called the denotation of  $p_\alpha$ ;
- ▶ the logical connectives  $\neg$ ,  $\vee$ , and  $\forall$  are always given the standard denotations;
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## Standard and Henkin Models

In a standard model  $M = \langle D, I \rangle$  we have

- ▶  $D_{\alpha \rightarrow \beta} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )

In a Henkin model  $M = \langle D, I \rangle$  we only require

- ▶  $D_{\alpha \rightarrow \beta} \subseteq \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )
- ▶ the valuation function  $\| \cdot \|^{M, g}$  from above is total (every term denotes)

Any standard model is obviously also a Henkin model.

We consider Henkin models in the remainder (for Henkin semantics we have semi-decidability, compactness and existence of countable models).

## A Remark on Equality

Primitive equality is not needed in HOL, since equality can be defined:

$$\doteq := \lambda x_\alpha \lambda y_\alpha \forall p_{\alpha \rightarrow o}. px \rightarrow py \quad (\text{Leibniz})$$

$$\div := \lambda x_\alpha \lambda y_\alpha \forall q_{\alpha \rightarrow \alpha \rightarrow o}. \forall z_\alpha (qzz) \rightarrow qxy \quad (\text{Andrews})$$

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## Variable Assignment

A variable assignment  $g$  maps variables  $x_\alpha$  to elements in  $D_\alpha$ .

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## Interpretation/Value of a HOL term

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1.  $\|P_\alpha\|^{M,g} = I(P_\alpha)$
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The value  $\|s_\alpha\|^{M,g}$  of a HOL term  $s_\alpha$  on a model  $M = \langle D, I \rangle$  under assignment  $g$  is an element  $d \in D_\alpha$  defined in the following way:

1.  $\|P_\alpha\|^{M,g} = I(P_\alpha)$
2.  $\|x_\alpha\|^{M,g} = g(x_\alpha)$
3.  $\|(s_{\alpha \rightarrow \beta} t_\alpha)_\beta\|^{M,g} = \|s_{\alpha \rightarrow \beta}\|^{M,g}(\|t_\alpha\|^{M,g})$
4.  $\|(\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta}\|^{M,g}$  = the function  $f$  from  $D_\alpha$  to  $D_\beta$  such that  $f(d) = \|s_\beta\|^{M,g[d/x_\alpha]}$  for all  $d \in D_\alpha$
5.  $I(\neg_{o \rightarrow o}) = \text{not} \in D_{o \rightarrow o}$  such that:  $\text{not}(F) = T$  and  $\text{not}(T) = F$
6.  $I(\vee_{o \rightarrow o \rightarrow o}) = \text{or} \in D_{o \rightarrow o \rightarrow o}$  such that:  $\text{or}(a, b) = T$  iff  $a = T$  or  $b = T$
7.  $\|(\vee_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha s_o))_o\|^{M,g} = T$  iff for all  $d \in D_\alpha$  we have  $\|s_o\|^{M,g[d/x_\alpha]} = T$

## Variable Assignment

A variable assignment  $g$  maps variables  $x_\alpha$  to elements in  $D_\alpha$ .

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4.  $\|(\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta}\|^{M,g} = \text{the function } f \text{ from } D_\alpha \text{ to } D_\beta \text{ such that } f(d) = \|s_\beta\|^{M,g[d/x_\alpha]} \text{ for all } d \in D_\alpha$
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7.  $I(\forall_{(\alpha \rightarrow o) \rightarrow o}) = \text{All} \in D_{(\alpha \rightarrow o) \rightarrow o}$  such that: for  $p \in D_{\alpha \rightarrow o}$  we have  $\text{All}(p) = T$  iff  $p(d) = T$  for all  $d \in D_\alpha$

## HOL: Semantics

### Truth in model / Validity

A formula  $s_o$  is true in model  $M$  under assignment  $g$  if and only if  $\|s_o\|^{M,g} = T$ ; this is also denoted as  $M, g \models s_o$ .

A formula  $s_o$  is called valid in  $M$  if and only if  $M, g \models s_o$  for all assignments  $g$ ; this is also denoted as  $M \models s_o$ .

A formula  $s_o$  is called valid, which we denote by  $\models s_o$ , if and only if  $M \models s_o$  for all  $M$ .

We define  $\models \varphi$ , where  $\varphi$  is a set of HOL formulas, if and only if  $\models s$  for all  $s \in \varphi$ .

### Logical Consequence

Let  $\varphi$  and  $\psi$  be set of HOL formulas. We define

$$\varphi \models \psi \quad (\text{logical consequence})$$

if and only if for each model  $M$  we have:

$$M \models \varphi \quad \text{implies} \quad M \models \psi$$

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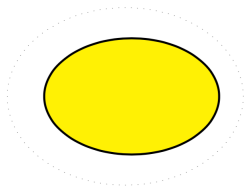
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# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



Standard Models  $\mathfrak{M}(\Sigma)$

## ■ Idea of Standard Semantics:

$\iota \longrightarrow \mathcal{D}_\iota$  (choose)

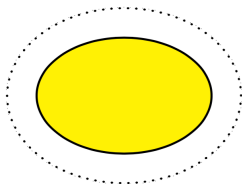
$\circ \longrightarrow \mathcal{D}_\circ = \{\mathbf{T}, \mathbf{F}\}$  (fixed)

$(\alpha \rightarrow \beta) \longrightarrow$

$\mathcal{D}_{\alpha \rightarrow \beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$  (fixed)

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



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## Henkin's Generalization:

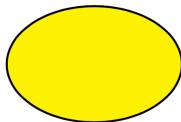
$\mathcal{D}_{\alpha \rightarrow \beta} \subseteq \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$  (choose)

but elements are still functions!

[Henkin-50]

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



Standard Models  $\mathfrak{M}(\Sigma)$

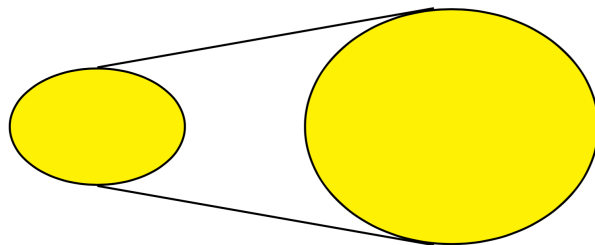
choose:  $\mathcal{D}_i$

fixed:  $\mathcal{D}_o, \mathcal{D}_{\alpha \rightarrow \beta}$ , functions



## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



Standard Models  $\mathfrak{SI}(\Sigma)$

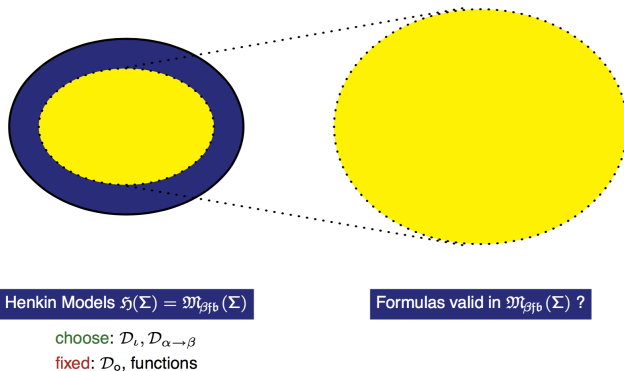
Formulas valid in  $\mathfrak{SI}(\Sigma)$

choose:  $\mathcal{D}_i$

fixed:  $\mathcal{D}_o, \mathcal{D}_{\alpha \rightarrow \beta}$ , functions

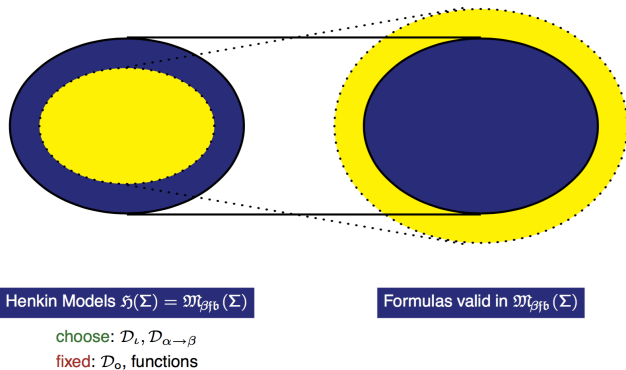
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(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



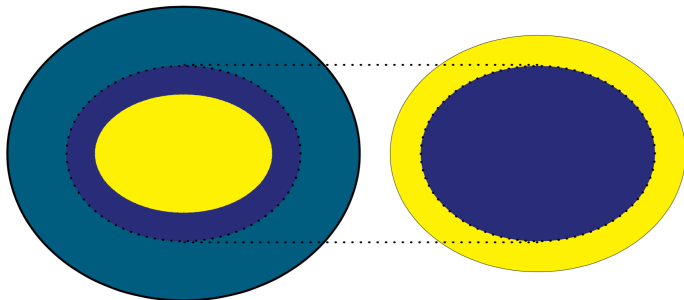
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## HOL: Semantics

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Non-Extensional Models  $\mathfrak{M}_\beta(\Sigma)$

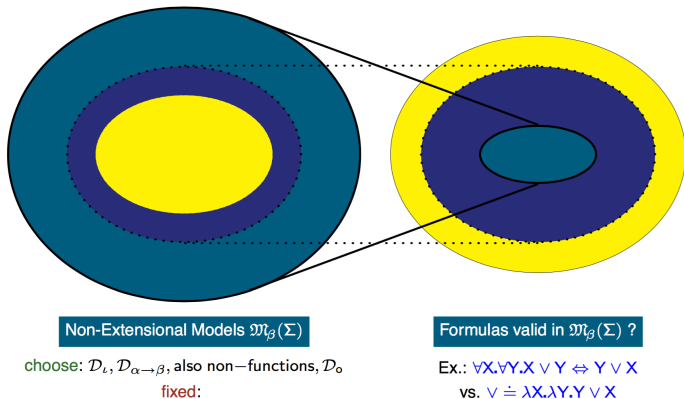
choose:  $\mathcal{D}_\iota, \mathcal{D}_{\alpha \rightarrow \beta}$ , also non-functions,  $\mathcal{D}_o$

fixed:

Formulas valid in  $\mathfrak{M}_\beta(\Sigma)$  ?

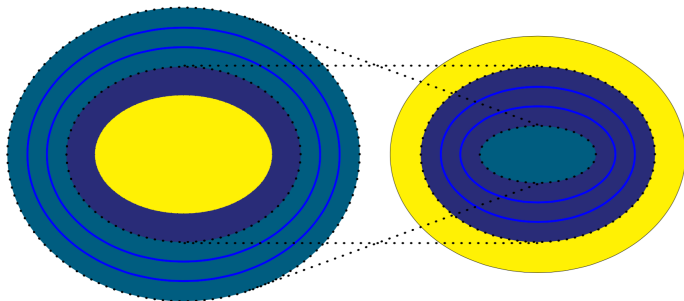
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(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



We additionally studied different model classes with 'varying degrees of extensionality'

$$\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$$

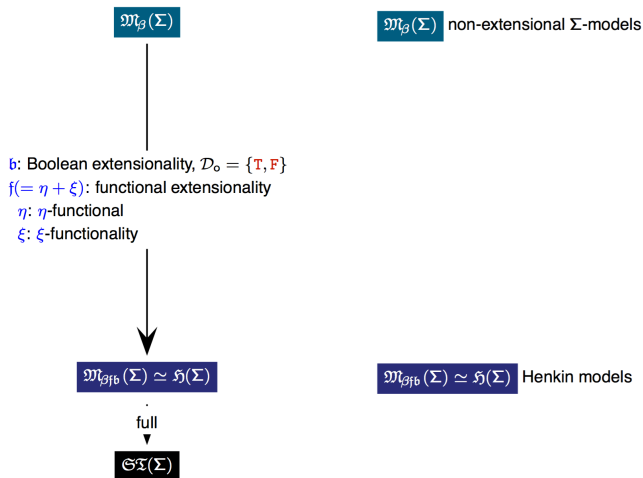
$$\forall X. \forall Y. X \vee Y \doteq Y \vee X$$

$$\lambda X. \lambda Y. X \vee Y \doteq \lambda X. \lambda Y. Y \vee X$$

$$\vee \doteq \lambda X. \lambda Y. Y \vee X$$

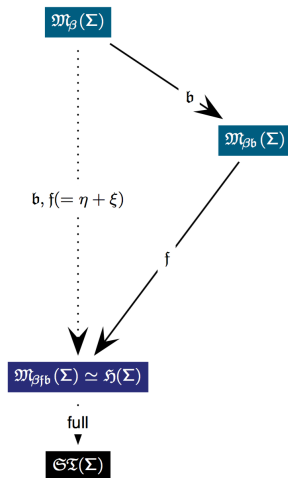
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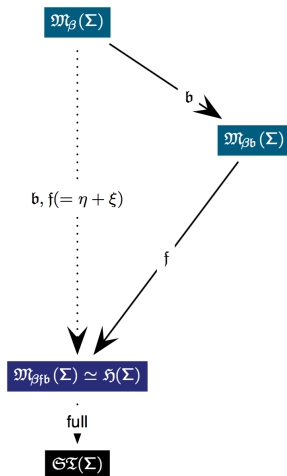
Motivation for  
Models without Functional Extensionality

- modeling programs:  
 $p_1 \neq p_2$  even if  $p_1@a = p_2@a$  for every  $a \in \mathcal{D}_\alpha$
- consider, e.g., run-time complexity:  
 $p_1 \leftarrow \lambda X.1$   
and  
 $p_2 \leftarrow \lambda X.1 + (X + 1)^2 - (X^2 + 2X + 1)$



# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

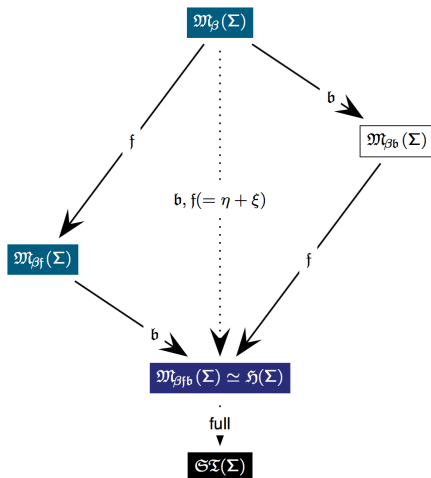


Motivation for  
Models without Boolean Extensionality?

- modeling of intensional concepts like 'knowledge', 'believe', etc.
- example:  
 $O := 2 + 2 = 4$   
 $F := \forall x, y, z, n > 2. x^n + y^n = z^n \Rightarrow x = y = z = 0$   
We want to model:  
 $O \Leftrightarrow F$  but  
 $\text{john\_knows}(F) \not\Leftrightarrow \text{john\_knows}(O)$
- if we have  $\mathcal{D}_o = \{\mathbf{T}, \mathbf{F}\}$  then  
 $O \Leftrightarrow F$  implies  $O = F$   
which also enforces  
 $\text{john\_knows}(F) \Leftrightarrow \text{john\_knows}(O)$

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

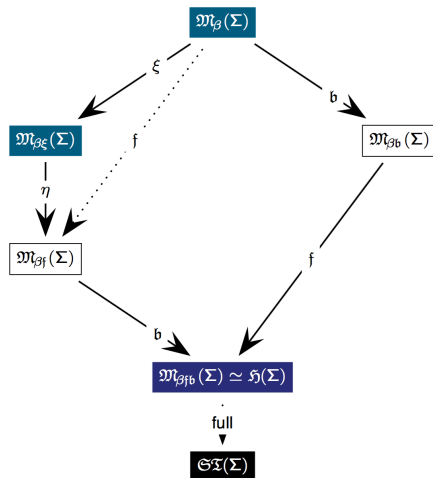


Models without  $\eta$

$$\mathcal{E}_{\varphi}(A) = \mathcal{E}_{\varphi}(A \downarrow_{\eta})$$

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

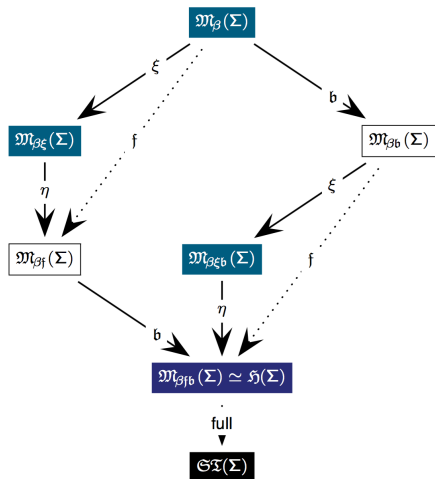


Models without  $\xi$

$$\mathcal{E}_\varphi(\lambda X_\alpha. M_\beta) = \mathcal{E}_\varphi(\lambda X_\alpha. N_\beta) \text{ iff } \mathcal{E}_{\varphi, [a/X]}(M) = \mathcal{E}_{\varphi, [a/X]}(N) \ (\forall a \in \mathcal{D}_\alpha)$$

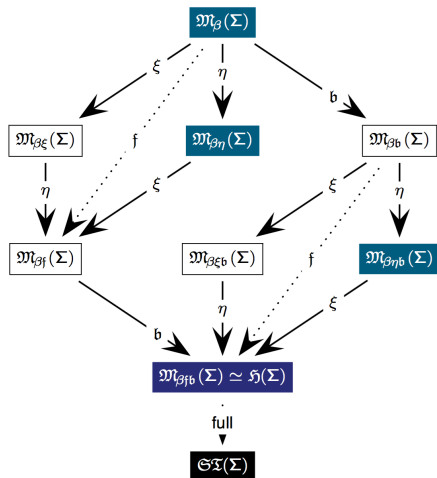
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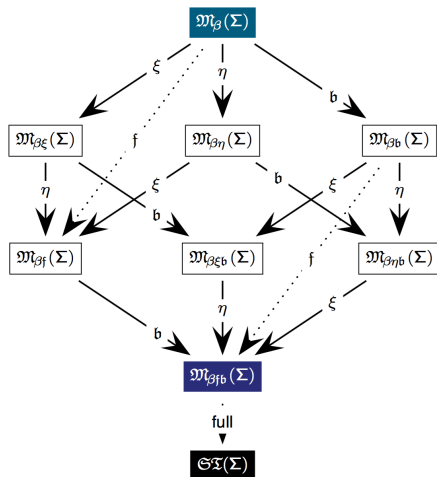
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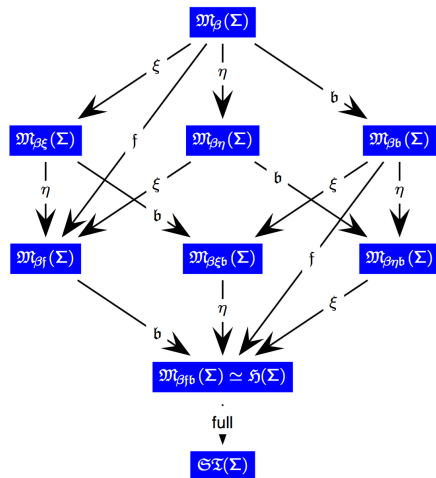
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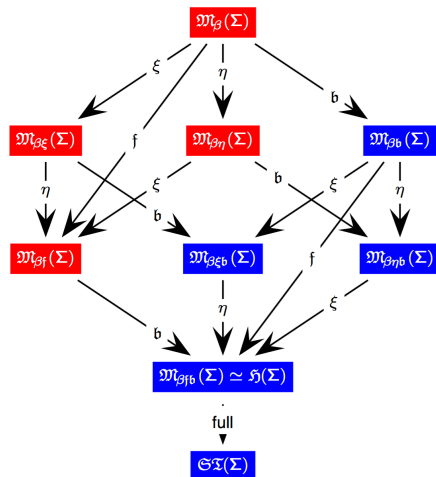


■  $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$

valid for all model classes

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



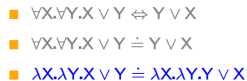
■  $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$

■  $\forall X. \forall Y. X \vee Y \doteq Y \vee X$

validity requires  $\flat$



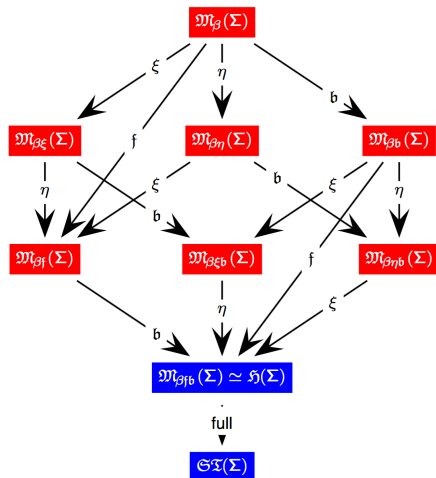
(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



C. Benzmüller and B. Woltzenlogel Paleo, 2015—Higher-Order Modal Logics

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

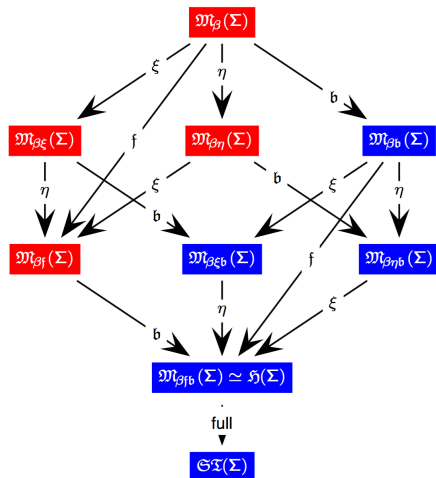


- $\forall X. \forall Y. X \vee Y \Leftrightarrow Y \vee X$
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- $\lambda X. \lambda Y. X \vee Y \doteq \lambda X. \lambda Y. Y \vee X$
- $\vee \doteq \lambda X. \lambda Y. Y \vee X$

validity requires b and f

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

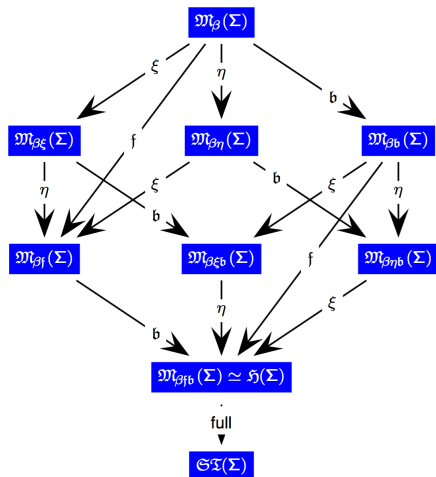


Examples requiring property  $b$

- $(p \ a_o) \wedge (p \ b_o) \Rightarrow (p \ (a \wedge b))$
- $(h_o \rightarrow_{\iota} ((h \top) \doteq (h \bot))) \doteq (h \bot)$

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

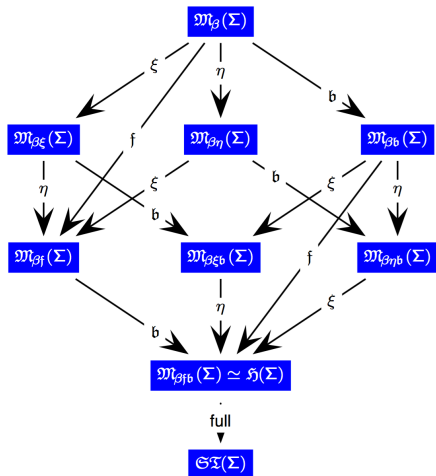


$\equiv$  is equivalence relation

- $\forall X_{\alpha}. X \equiv X$
- $\forall X_{\alpha}, Y_{\alpha}. X \equiv Y \supset Y \equiv X$
- $\forall X_{\alpha}, Y_{\alpha}, Z_{\alpha}. (X \equiv Y \wedge Y \equiv Z) \supset X \equiv Z$

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



$\equiv$  is equivalence relation

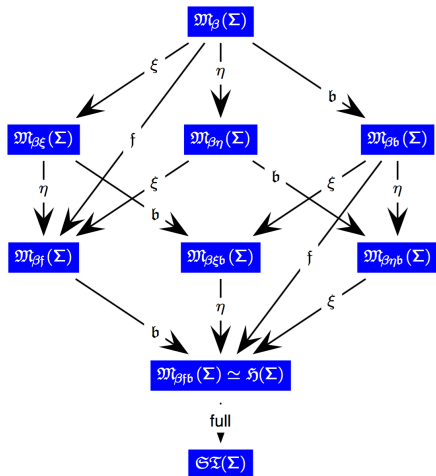
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$\equiv$  is congruence relation

- $\forall X_{\alpha}, Y_{\alpha}, F_{\alpha\alpha}. X \equiv Y \supset (FX) \equiv (FY)$
- $\forall X_{\alpha}, Y_{\alpha}, P_{\alpha\alpha}. X \equiv Y \wedge (PX) \supset (PY)$

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



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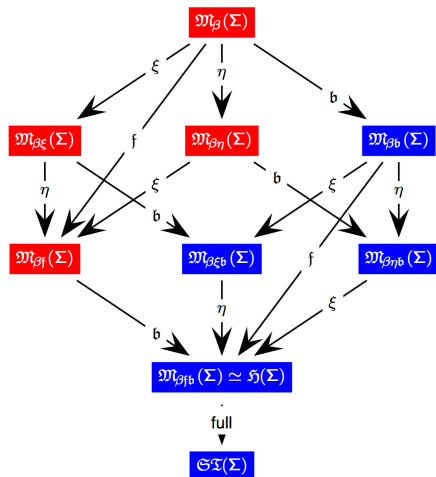
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Trivial directions of Boolean and functional extensionality

- $\forall A_o, B_o. A \equiv B \supset (A \leftrightarrow B)$
- $\forall F_{\beta\alpha}, G_{\beta\alpha}. F \equiv G \supset (\forall X_\alpha. FX \equiv GX)$

# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



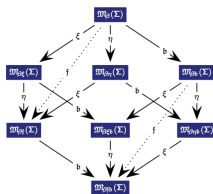
Non-trivial direction of Boolean extensionality

■  $\forall A_o, B_o. (A \Leftrightarrow B) \supset A \stackrel{*}{=} B$

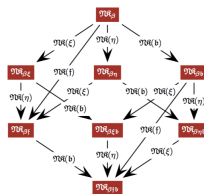
# HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

Semantics:  
Model Classes (Extensionality)



Reference Calculi:  
ND (and others)

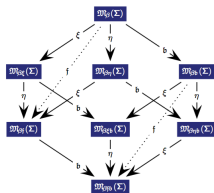




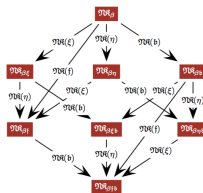
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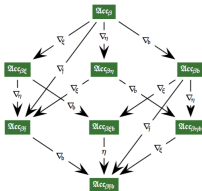
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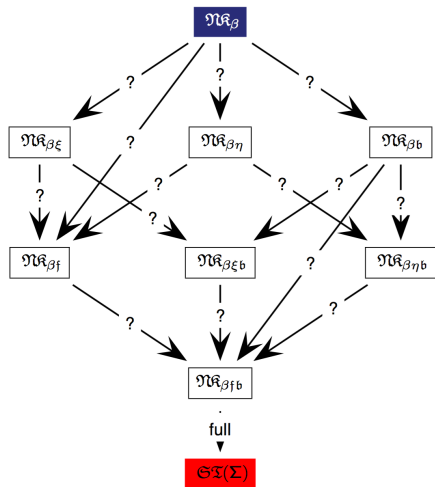


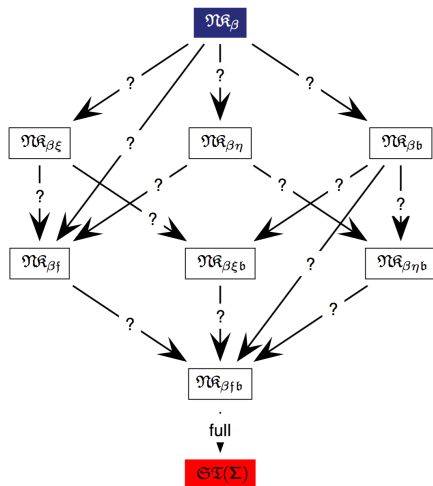
Reference Calculi:  
ND (and others)



Abstract Consistency / Unifying Principle:  
Extensions of Smullyan-63 and Andrews-71







Base Calculus  $\mathcal{N}\mathcal{R}_\beta$

—  $\mathcal{N}\mathcal{R}(Hyp)$  —  $\mathcal{N}\mathcal{R}(\beta)$

—  $\mathcal{N}\mathcal{R}(\neg I)$  —  $\mathcal{N}\mathcal{R}(\neg E)$

—  $\mathcal{N}\mathcal{R}(\vee I_L)$  —  $\mathcal{N}\mathcal{R}(\vee I_R)$

—  $\mathcal{N}\mathcal{R}(\vee E)$

—  $\mathcal{N}\mathcal{R}(III)^w$

—  $\mathcal{N}\mathcal{R}(IE)$  —  $\mathcal{N}\mathcal{R}(Contr)$

$\mathcal{N}\mathcal{A}_\beta$ :

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathcal{N}\mathcal{A}(Hyp)$$

$$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$$

—

—

$\mathfrak{N}\mathfrak{A}_\beta$ :

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{N}\mathfrak{A}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{N}\mathfrak{A}(\beta)$$

$$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$$

$\mathcal{N}\mathcal{R}_\beta$ :

$$\begin{array}{c}
 \frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathcal{N}\mathcal{R}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathcal{N}\mathcal{R}(\beta) \\
 \\
 \frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg \mathbf{A}} \mathcal{N}\mathcal{R}(\neg I)
 \end{array}$$

$$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$$

$\mathcal{N}\mathcal{A}_\beta$ :

|                                                                                                                |                                                                                                                           |
|----------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------|
| $\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathcal{N}\mathcal{A}(Hyp)$                               | $\frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathcal{N}\mathcal{A}(\beta)$ |
| $\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg \mathbf{A}} \mathcal{N}\mathcal{A}(\neg I)$ | $\frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathcal{N}\mathcal{A}(\neg E)$  |

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

$\mathcal{N}\mathcal{R}_\beta$ :

|                                                                                                                |                                                                                                                           |
|----------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------|
| $\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathcal{N}\mathcal{R}(Hyp)$                               | $\frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathcal{N}\mathcal{R}(\beta)$ |
| $\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg \mathbf{A}} \mathcal{N}\mathcal{R}(\neg I)$ | $\frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathcal{N}\mathcal{R}(\neg E)$  |
| $\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{R}(\vee I_L)$       |                                                                                                                           |

$$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$$



$\mathcal{N}\mathcal{A}_\beta$ :

|                                                                                                                |                                                                                                                           |
|----------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------|
| $\frac{\mathbf{A} \in \Phi}{\Phi \vdash \mathbf{A}} \mathcal{N}\mathcal{A}(Hyp)$                               | $\frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{B}} \mathcal{N}\mathcal{A}(\beta)$ |
| $\frac{\Phi * \mathbf{A} \vdash \mathbf{F}_\circ}{\Phi \vdash \neg \mathbf{A}} \mathcal{N}\mathcal{A}(\neg I)$ | $\frac{\Phi \vdash \neg \mathbf{A} \quad \Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{C}} \mathcal{N}\mathcal{A}(\neg E)$  |
| $\frac{\Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{A}(\vee I_L)$       | $\frac{\Phi \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{A}(\vee I_R)$                  |

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

$\mathcal{N}\mathcal{R}_\beta$ :

$$\begin{array}{c}
 \frac{\mathbf{A} \in \Phi}{\Phi \vdash \mathbf{A}} \mathcal{N}\mathcal{R}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{B}} \mathcal{N}\mathcal{R}(\beta) \\
 \\
 \frac{\Phi * \mathbf{A} \vdash \mathbf{F}_o}{\Phi \vdash \neg \mathbf{A}} \mathcal{N}\mathcal{R}(\neg I) \qquad \frac{\Phi \vdash \neg \mathbf{A} \quad \Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{C}} \mathcal{N}\mathcal{R}(\neg E) \\
 \\
 \frac{\Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{R}(\vee I_L) \qquad \frac{\Phi \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{R}(\vee I_R) \\
 \\
 \frac{\Phi \vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \vdash \mathbf{C} \quad \Phi * \mathbf{B} \vdash \mathbf{C}}{\Phi \vdash \mathbf{C}} \mathcal{N}\mathcal{R}(\vee E)
 \end{array}$$

$$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$$

$\mathfrak{N}\mathfrak{A}_\beta$ :

$$\begin{array}{c}
 \frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{N}\mathfrak{A}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{N}\mathfrak{A}(\beta) \\
 \\
 \frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}} \mathfrak{N}\mathfrak{A}(\neg I) \qquad \frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathfrak{N}\mathfrak{A}(\neg E) \\
 \\
 \frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{N}\mathfrak{A}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{N}\mathfrak{A}(\vee I_R) \\
 \\
 \frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \mathfrak{N}\mathfrak{A}(\vee E) \\
 \\
 \frac{\Phi \Vdash \mathbf{G}_{w_\alpha} \quad w \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}} \mathfrak{N}\mathfrak{A}(III)^w
 \end{array}$$

$$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$$

$\mathfrak{N}\mathfrak{A}_\beta$ :

|                                                                                                                                                                                              |                                                                                                                             |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------|
| $\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{N}\mathfrak{A}(Hyp)$                                                                                                           | $\frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{N}\mathfrak{A}(\beta)$ |
| $\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}} \mathfrak{N}\mathfrak{A}(\neg I)$                                                                                 | $\frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathfrak{N}\mathfrak{A}(\neg E)$  |
| $\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{N}\mathfrak{A}(\vee I_L)$                                                                                   | $\frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{N}\mathfrak{A}(\vee I_R)$                  |
| $\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \mathfrak{N}\mathfrak{A}(\vee E)$ |                                                                                                                             |
| $\frac{\Phi \Vdash \mathbf{G}w_\alpha \quad w \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}} \mathfrak{N}\mathfrak{A}(III)^w$                                                     |                                                                                                                             |
| $\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}} \mathfrak{N}\mathfrak{A}(IE)$                                                                                    |                                                                                                                             |

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

$\mathcal{N}\mathcal{R}_\beta$ :

|                                                                                                                                                                                            |                                                                                                                           |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------|
| $\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathcal{N}\mathcal{R}(Hyp)$                                                                                                           | $\frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathcal{N}\mathcal{R}(\beta)$ |
| $\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg \mathbf{A}} \mathcal{N}\mathcal{R}(\neg I)$                                                                             | $\frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathcal{N}\mathcal{R}(\neg E)$  |
| $\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{R}(\vee I_L)$                                                                                   | $\frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathcal{N}\mathcal{R}(\vee I_R)$                  |
| $\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \mathcal{N}\mathcal{R}(\vee E)$ |                                                                                                                           |
| $\frac{\Phi \Vdash \mathbf{G}w_\alpha \quad w \text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}} \mathcal{N}\mathcal{R}(III)^w$                                                     |                                                                                                                           |
| $\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}} \mathcal{N}\mathcal{R}(IIE)$                                                                                   | $\frac{\Phi * \neg \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \mathbf{A}} \mathcal{N}\mathcal{R}(Contr)$             |

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

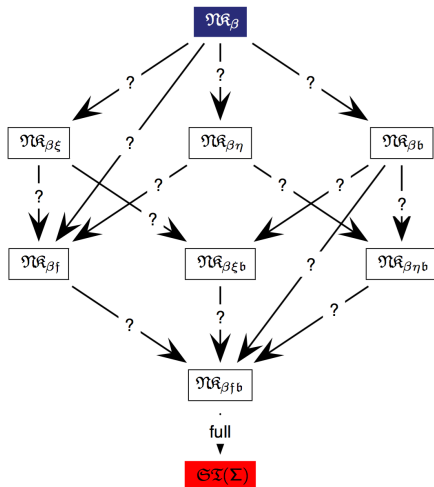
Inference rules for  $\mathfrak{N}\mathfrak{R}_\beta$  (for richer signatures)

$$\begin{array}{c}
 \frac{\Phi \vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \vdash \mathbf{A}} \mathfrak{N}\mathfrak{R}(\wedge E_L) \quad \frac{\Phi \vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \vdash \mathbf{B}} \mathfrak{N}\mathfrak{R}(\wedge E_R) \quad \frac{\Phi \vdash \mathbf{A} \quad \Phi \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \wedge \mathbf{B}} \mathfrak{N}\mathfrak{R}(\wedge I) \\
 \\
 \frac{\Phi \vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{B}} \mathfrak{N}\mathfrak{R}(\Rightarrow E) \quad \frac{\Phi, \mathbf{A} \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \Rightarrow \mathbf{B}} \mathfrak{N}\mathfrak{R}(\Rightarrow I) \\
 \\
 \frac{\Phi \vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \vdash \Sigma^\alpha \mathbf{G}} \mathfrak{N}\mathfrak{R}(\Sigma I) \quad \frac{\Phi \vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}_{\mathbf{W}_\alpha} \vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \vdash \mathbf{C}} \mathfrak{N}\mathfrak{R}(\Sigma E) \\
 \\
 \frac{\Phi \vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \vdash \mathbf{A}[\mathbf{T}]}{\Phi \vdash \mathbf{A}[\mathbf{W}]} \mathfrak{N}\mathfrak{R}(= Subst) \quad \frac{}{\Phi \vdash \mathbf{A} = \mathbf{A}} \mathfrak{N}\mathfrak{R}(= Refl)
 \end{array}$$

Inference rules for  $\mathfrak{N}\mathfrak{R}_\beta$  (for richer signatures)

$$\begin{array}{c}
 \frac{\Phi \vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \vdash \mathbf{A}} \mathfrak{N}\mathfrak{R}(\wedge E_L) \quad \frac{\Phi \vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \vdash \mathbf{B}} \mathfrak{N}\mathfrak{R}(\wedge E_R) \quad \frac{\Phi \vdash \mathbf{A} \quad \Phi \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \wedge \mathbf{B}} \mathfrak{N}\mathfrak{R}(\wedge I) \\
 \\
 \frac{\Phi \vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \vdash \mathbf{A}}{\Phi \vdash \mathbf{B}} \mathfrak{N}\mathfrak{R}(\Rightarrow E) \quad \frac{\Phi, \mathbf{A} \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \Rightarrow \mathbf{B}} \mathfrak{N}\mathfrak{R}(\Rightarrow I) \\
 \\
 \frac{\Phi \vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \vdash \Sigma^\alpha \mathbf{G}} \mathfrak{N}\mathfrak{R}(\Sigma I) \quad \frac{\Phi \vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathbf{W}_\alpha \vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \vdash \mathbf{C}} \mathfrak{N}\mathfrak{R}(\Sigma E) \\
 \\
 \frac{\Phi \vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \vdash \mathbf{A}[\mathbf{T}]}{\Phi \vdash \mathbf{A}[\mathbf{W}]} \mathfrak{N}\mathfrak{R}(= Subst) \quad \frac{}{\Phi \vdash \mathbf{A} = \mathbf{A}} \mathfrak{N}\mathfrak{R}(= Refl)
 \end{array}$$

Here: we define logical constants  $\wedge, \Rightarrow, \Sigma$ , etc. in terms of  $\neg, \vee, \Box$  as usual and strictly use Leibniz equality instead of primitive equality; then the above rules are not needed



Base Calculus  $\mathcal{NL}_\beta$

—  $\mathcal{NL}(Hyp)$  —  $\mathcal{NL}(\beta)$

—  $\mathcal{NL}(\neg I)$  —  $\mathcal{NL}(\neg E)$

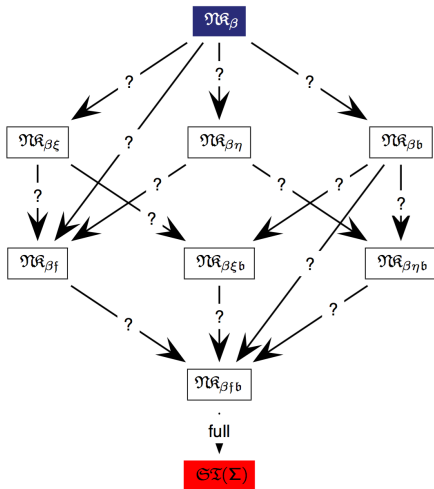
—  $\mathcal{NL}(\vee I_L)$  —  $\mathcal{NL}(\vee I_R)$

—  $\mathcal{NL}(\vee E)$

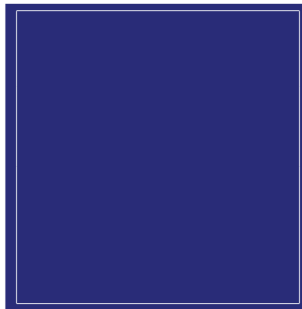
—  $\mathcal{NL}(\Pi I)^w$

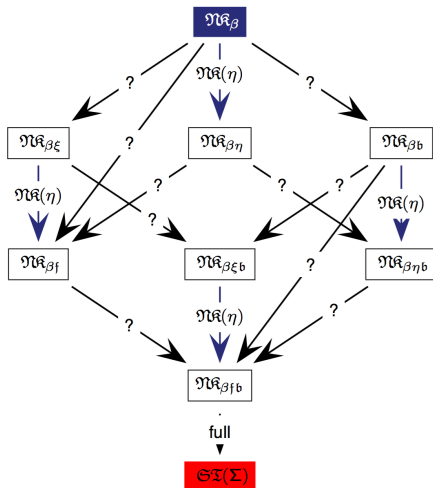
—  $\mathcal{NL}(\Pi E)$  —  $\mathcal{NL}(Contr)$





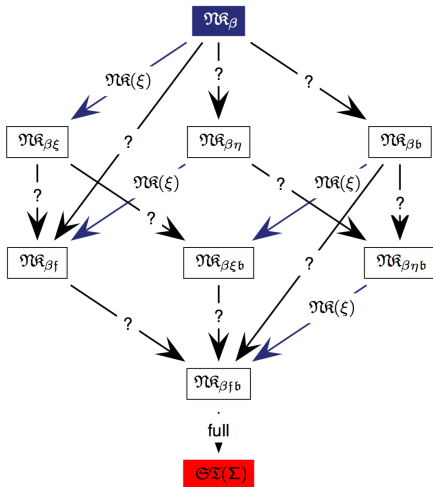
## Optional Extensionality Rules





Optional Extensionality Rules

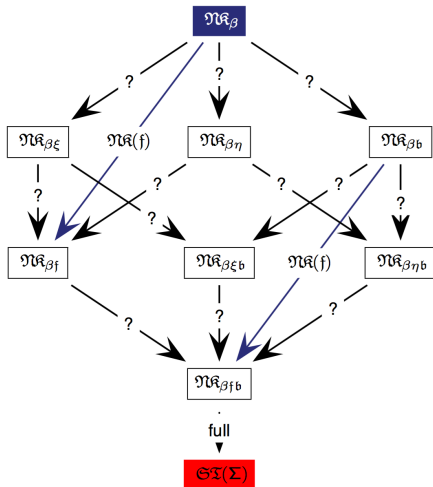
$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \vdash A}{\Phi \vdash B} \mathcal{M}(\eta)$$



## Optional Extensionality Rules

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \Vdash A}{\Phi \vdash B} \mathcal{NL}(\eta)$$

$$\frac{\Phi \vdash \forall x_\alpha. M \stackrel{\beta}{=} N}{\Phi \vdash (\lambda x_\alpha. M) \stackrel{\beta\alpha}{=} (\lambda x_\alpha. N)} \mathcal{NL}(\xi)$$

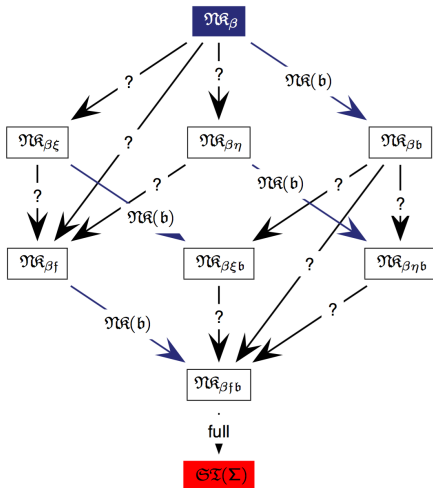


## Optional Extensionality Rules

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \vdash A}{\Phi \vdash B} \mathfrak{N}\mathfrak{A}(\eta)$$

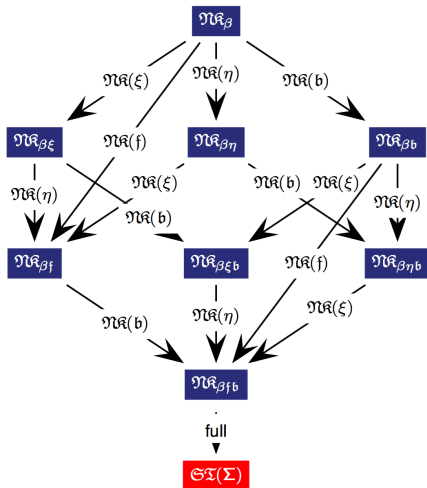
$$\frac{\Phi \vdash \forall x_\alpha. M \stackrel{\beta}{=} N}{\Phi \vdash (\lambda x_\alpha. M) \stackrel{\beta\alpha}{=} (\lambda x_\alpha. N)} \mathfrak{N}\mathfrak{A}(\xi)$$

$$\frac{\Phi \vdash \forall x_\alpha. Gx \stackrel{\beta}{=} Hx}{\Phi \vdash G \stackrel{\beta\alpha}{=} H} \mathfrak{N}\mathfrak{A}(f)$$



## Optional Extensionality Rules

$$\begin{array}{c}
 \frac{A \stackrel{\beta\eta}{=} B \quad \Phi \vdash A}{\Phi \vdash B} \mathcal{NL}(\eta) \\
 \\
 \frac{\Phi \vdash \forall x_\alpha. M \stackrel{\beta}{=} N}{\Phi \vdash (\lambda x_\alpha. M) \stackrel{\beta\alpha}{=} (\lambda x_\alpha. N)} \mathcal{NL}(\xi) \\
 \\
 \frac{\Phi \vdash \forall x_\alpha. Gx \stackrel{\beta}{=} Hx}{\Phi \vdash G \stackrel{\beta\alpha}{=} H} \mathcal{NL}(f) \\
 \\
 \frac{\Phi * A \Vdash B \quad \Phi * B \vdash A}{\Phi \vdash A \stackrel{\circ}{=} B} \mathcal{NL}(b)
 \end{array}$$



## Base Calculus $\mathcal{N}\mathcal{A}_\beta$

- $\mathcal{N}\mathcal{A}(Hyp)$  —  $\mathcal{N}\mathcal{A}(\beta)$
- $\mathcal{N}\mathcal{A}(\neg I)$  —  $\mathcal{N}\mathcal{A}(\neg E)$
- $\mathcal{N}\mathcal{A}(\vee I_L)$  —  $\mathcal{N}\mathcal{A}(\vee I_R)$
- $\mathcal{N}\mathcal{A}(\vee E)$
- $\mathcal{N}\mathcal{A}(\Pi)^w$
- $\mathcal{N}\mathcal{A}(\Pi E)$  —  $\mathcal{N}\mathcal{A}(Contr)$

## Optional Extensionality Rules

- $\mathcal{N}\mathcal{A}(\eta)$  —  $\mathcal{N}\mathcal{A}(\xi)$
- $\mathcal{N}\mathcal{A}(f)$  —  $\mathcal{N}\mathcal{A}(b)$

Derivation of:

$$(A \dot{=}^{\alpha} A)$$

$$\frac{}{\Vdash_{\mathfrak{M}_\beta} (A \dot{=}^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA)))} \mathfrak{N}(\text{III})$$

Derivation of:

$$(A \dot{=}^\alpha A)$$



$$\frac{\frac{}{\Vdash_{\mathfrak{K}_\beta} ((\lambda P(\neg(PA) \vee (PA)))q)} \mathfrak{N}(\beta)}{\Vdash_{\mathfrak{K}_\beta} (A \dot{=}^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA)))} \mathfrak{N}(III)$$

Derivation of:

$$(A \dot{=}^\alpha A)$$

$$\begin{array}{c}
 \frac{}{\vdash_{\mathfrak{N}_\beta} \neg(qA) \vee (qA)} \mathfrak{N}(Contr) \\
 \frac{}{\vdash_{\mathfrak{N}_\beta} ((\lambda P(\neg(PA) \vee (PA)))q)} \mathfrak{N}(\beta) \\
 \hline
 \vdash_{\mathfrak{N}_\beta} (A \doteq^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA))) \mathfrak{N}(III)
 \end{array}$$

Derivation of:

$$(A \doteq^\alpha A)$$

$$\begin{array}{c}
 \frac{}{\vdash_{\mathfrak{K}_\beta} \Phi^1 := \{\neg(\neg(qA) \vee (qA))\} \vdash_{\mathfrak{K}_\beta} \mathbf{F}_o} \mathfrak{K}(\neg E) \\
 \frac{}{\vdash_{\mathfrak{K}_\beta} \neg(qA) \vee (qA)} \mathfrak{K}(Contr) \\
 \frac{}{\vdash_{\mathfrak{K}_\beta} ((\lambda P(\neg(PA) \vee (PA)))q)} \mathfrak{K}(\beta) \\
 \frac{}{\vdash_{\mathfrak{K}_\beta} (A \doteq^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA)))} \mathfrak{K}(III)
 \end{array}$$

Derivation of:

$$(A \doteq^\alpha A)$$

$$\begin{array}{c}
 \Phi^1 \Vdash_{\mathfrak{N}\mathfrak{R}_\beta} \neg(qA) \vee (qA) \\
 \hline
 \Phi^1 := \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{N}\mathfrak{R}_\beta} \mathbf{F}_o \quad \mathfrak{N}\mathfrak{R}(\neg E) \\
 \hline
 \Vdash_{\mathfrak{N}\mathfrak{R}_\beta} \neg(qA) \vee (qA) \quad \mathfrak{N}\mathfrak{R}(Contr) \\
 \hline
 \Vdash_{\mathfrak{N}\mathfrak{R}_\beta} ((\lambda P(\neg(PA) \vee (PA)))q) \quad \mathfrak{N}\mathfrak{R}(\beta) \\
 \hline
 \Vdash_{\mathfrak{N}\mathfrak{R}_\beta} (A \doteq^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA))) \quad \mathfrak{N}\mathfrak{R}(III)
 \end{array}$$

Derivation of:

$$(A \doteq^\alpha A)$$

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{\frac{\frac{\Phi^1 \Vdash_{\mathfrak{K}_\beta} \neg(qA) \vee (qA)}{\Phi^1 \Vdash_{\mathfrak{K}_\beta} \{\neg(\neg(qA) \vee (qA))\}} \mathfrak{N}(Hyp)}{\Phi^1 \Vdash_{\mathfrak{K}_\beta} \neg(qA) \vee (qA)} \mathfrak{N}(\neg E)}{\Phi^1 := \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{K}_\beta} \mathbf{F}_o} \mathfrak{N}(Contr)}}{\Vdash_{\mathfrak{K}_\beta} \neg(qA) \vee (qA)} \mathfrak{N}(\beta)}}{\Vdash_{\mathfrak{K}_\beta} ((\lambda P(\neg(PA) \vee (PA)))q)} \mathfrak{N}(III)}}{\Vdash_{\mathfrak{K}_\beta} (A \doteq^\alpha A) := \Pi^\alpha(\lambda P(\neg(PA) \vee (PA)))}
 \end{array}$$

Derivation of:

$$(A \doteq^\alpha A)$$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{M}_\beta} \neg p \vee p$$

resp.  $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{M}_\beta} \neg(qA) \vee (qA)$

$$\frac{}{\{\neg(\neg p \vee p)\} \Vdash_{\mathcal{M}_\beta} \neg p \vee p} \mathcal{NI}(\vee I_L)$$

Derivation of:

$$\begin{aligned} & \{\neg(\neg p \vee p)\} \Vdash_{\mathcal{M}_\beta} \neg p \vee p \\ \text{resp. } & \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathcal{M}_\beta} \neg(qA) \vee (qA) \end{aligned}$$

$$\frac{\frac{\frac{}{\{ \neg(\neg p \vee p) \} \Vdash_{\mathcal{M}_\beta} \neg p} \text{NI}(\neg I)}{\{ \neg(\neg p \vee p) \} \Vdash_{\mathcal{M}_\beta} \neg p \vee p} \text{NI}(\vee I_L)}{\{ \neg(\neg p \vee p) \} \Vdash_{\mathcal{M}_\beta} \neg p \vee p}$$

Derivation of:

$$\{ \neg(\neg p \vee p) \} \Vdash_{\mathcal{M}_\beta} \neg p \vee p$$

resp.  $\{ \neg(\neg(qA) \vee (qA)) \} \Vdash_{\mathcal{M}_\beta} \neg(qA) \vee (qA)$



$$\begin{array}{c}
 \hline
 \Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{N}\mathfrak{A}_\beta} \mathbf{F}_o \\
 \hline
 \{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{N}\mathfrak{A}_\beta} \neg p \quad \mathfrak{N}\mathfrak{A}(\neg I) \\
 \hline
 \{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{N}\mathfrak{A}_\beta} \neg p \vee p \quad \mathfrak{N}\mathfrak{A}(\vee I_L)
 \end{array}$$

Derivation of:

$$\begin{array}{c}
 \{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{N}\mathfrak{A}_\beta} \neg p \vee p \\
 \text{resp. } \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{N}\mathfrak{A}_\beta} \neg(qA) \vee (qA)
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg(\neg p \vee p)} \mathfrak{N}\mathfrak{A}(Hyp) \\
 \frac{}{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{A}_\beta} \mathbf{F}_o} \mathfrak{N}\mathfrak{A}(\neg E) \\
 \frac{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{A}_\beta} \mathbf{F}_o}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p} \mathfrak{N}\mathfrak{A}(\neg I) \\
 \frac{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p \vee p} \mathfrak{N}\mathfrak{A}(\vee I_L)
 \end{array}$$

Derivation of:

$$\begin{array}{c}
 \{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p \vee p \\
 \text{resp. } \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{A}_\beta} \neg(qA) \vee (qA)
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg(\neg p \vee p)} \mathfrak{N}\mathfrak{I}(Hyp) \quad \frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg p \vee p} \mathfrak{N}\mathfrak{I}(\vee I_R) \\
 \hline
 \frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg(\neg p \vee p)} \mathfrak{N}\mathfrak{I}(\neg E) \\
 \frac{}{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{A}_\beta} \mathbf{F}_o} \mathfrak{N}\mathfrak{I}(\neg I) \\
 \frac{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p \vee p} \mathfrak{N}\mathfrak{I}(\vee I_L)
 \end{array}$$

Derivation of:

$$\begin{array}{c}
 \{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p \vee p \\
 \text{resp. } \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{A}_\beta} \neg(qA) \vee (qA)
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{\frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg(\neg p \vee p)}{\text{Hyp}}}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg(\neg p \vee p)} \quad \frac{\frac{\frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} p}}{\text{Hyp}} \quad \frac{}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg p \vee p} \text{Hyp}}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg p \vee p} \text{Hyp}}{\Phi^2 \Vdash_{\mathfrak{A}_\beta} \neg(\neg p \vee p)} \text{Hyp} \\
 \frac{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{A}_\beta} \mathbf{F}_o}{\frac{\frac{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p}{\text{Hyp}} \quad \frac{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p \vee p}{\text{Hyp}}}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p} \text{Hyp}} \text{Hyp}
 \end{array}$$

Derivation of:

$$\begin{array}{c}
 \{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{A}_\beta} \neg p \vee p \\
 \text{resp. } \{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{A}_\beta} \neg(qA) \vee (qA)
 \end{array}$$

# HOL: Sequent Calculi

One-sided sequent calculus  $\mathcal{G}_{\beta\text{fb}}$  [BenzmüllerBrownKohlhase, LMCS, 2009]

( $\Delta$ : finite sets of  $\beta$ -normal closed formulas,  $\Delta * \mathbf{A}$  stands for  $\Delta \cup \{\mathbf{A}\}$ ,  
 $\text{cwff}_\alpha$ : set of closed terms of type  $\alpha$ ,  $\doteq$  abbreviates Leibniz equality):

$$\begin{array}{l}
 \text{Base Rules} \quad \frac{\mathbf{A} \text{ atomic (\& } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg \mathbf{A}} \mathcal{G}(\text{init}) \quad \frac{\Delta * \mathbf{A}}{\Delta * \neg \neg \mathbf{A}} \mathcal{G}(\neg) \quad \frac{\Delta * \neg \mathbf{A} \quad \Delta * \neg \mathbf{B}}{\Delta * \neg (\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_-) \\
 \\
 \frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_+) \quad \frac{\Delta * \neg (\mathbf{A} \mathbf{C}) \downarrow_\beta \quad \mathbf{C} \in \text{cwff}_\alpha}{\Delta * \neg \Pi^\alpha \mathbf{A}} \mathcal{G}(\Pi^C_-) \quad \frac{\Delta * (\mathbf{A} \mathbf{C}) \downarrow_\beta \quad c_{\alpha}^{\text{new}}}{\Delta * \Pi^\alpha \mathbf{A}} \mathcal{G}(\Pi^C_+) \\
 \\
 \text{Full Extensionality} \quad \frac{\Delta * (\forall X_\alpha. \mathbf{A} X \doteq^\beta \mathbf{B} X) \downarrow_\beta}{\Delta * (\mathbf{A} \doteq^{\alpha \rightarrow \beta} \mathbf{B})} \mathcal{G}(\text{f}) \quad \frac{\Delta * \neg \mathbf{A} * \mathbf{B} \quad \Delta * \neg \mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^\circ \mathbf{B})} \mathcal{G}(\text{b}) \\
 \\
 \text{Initial. and Decomp. of Leibniz Equality} \quad \frac{\Delta * (\mathbf{A} \doteq^\circ \mathbf{B}) \quad \mathbf{A}, \mathbf{B} \text{ atomic}}{\Delta * \neg \mathbf{A} * \mathbf{B}} \mathcal{G}(\text{Init}^\doteq) \\
 \\
 \frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \dots \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad n \geq 1, \beta \in \{\circ, \iota\}, h_{\alpha^n \rightarrow \beta} \in \Sigma}{\Delta * (h\overline{\mathbf{A}}^n \doteq^\beta h\overline{\mathbf{B}}^n)} \mathcal{G}(d)
 \end{array}$$

## Soundness, Completeness and Cut-elimination

$$\text{iff } \mathbf{A}x \models_{\text{Henkin}}^{\text{HOL}} \varphi_{\iota \rightarrow o} \quad \text{iff } \mathbf{A}x \vdash_{\text{cut-free}}^{\text{HOL}} \varphi_{\iota \rightarrow o}$$



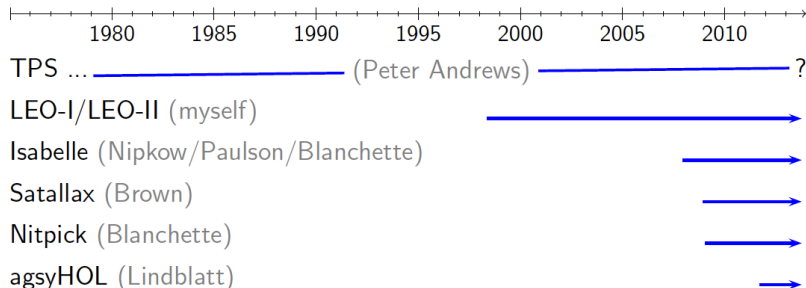
## Higher-Order Automated Theorem Provers (HOL-ATPs)

## EU FP7 Project THFTPTP

- ▶ Collaboration with Geoff Sutcliffe and others (Chad Brown, Florian Rabe, Nik Sultana, Jasmin Blanchette, Frank Theiss, ...)
- ▶ Results
  - ▶ THF0 syntax for HOL (with Choice; Henkin Semantics)
  - ▶ library with example problems (e.g. entire TPS library) and results
  - ▶ international CASC competition for HOL-ATP
  - ▶ online access to provers
  - ▶ various tools

More information: [\[SutcliffeBenzmüller, J.FormalizedReasoning, 2010\]](http://cordis.europa.eu/result/report/rcn/45614_en.html)  
[http://cordis.europa.eu/result/report/rcn/45614\\_en.html](http://cordis.europa.eu/result/report/rcn/45614_en.html)

## Automated Theorem Provers and Model Finders for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic  
Automate other logics (& combinations) via semantic embeddings  
— **HOL-P** becomes a **Universal Reasoner** —



- ▶ **TPS:** mating method, schedules many different modes (sets of flag settings)
- ▶ **LEO-I/II:** resolution based, cooperation with FOL-ATP (E), very few modes
- ▶ **Isabelle:** schedules various tactics and external provers
- ▶ **Satallax:** instantiation and tableaux based, cooperation with SAT solver (MiniSat) and FOL-ATP (E), schedules many different modes
- ▶ **Nitpick:** (counter-)model finder, cooperation with FOL constraint solver
- ▶ **agsyHOL:** generic lazy narrowing search algorithm with backtracking
- ▶ **coqATP:** implements (non-inductive) part of Coq's logic (calculus of constructions), adds axioms like excluded middle and extensionality

**Interactive Proof Assistants:** Isabelle/HOL, HOL, HOL-light, PVS, OMEGA, . . . , Automath, Nuprl, LEGO, Coq, Coq, Agda, . . .

**Logical Frameworks:** Elf, Twelf, Beluga, . . .

**Other HOL-ATPs:** OMEGA,  $\lambda$ Clam, Otter- $\lambda$ , Watson, . . .

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**Logical Frameworks:** Elf, Twelf, Beluga, . . .

**Other HOL-ATPs:** OMEGA,  $\lambda$ Clam, Otter- $\lambda$ , Watson, . . .

## Demo: TPTP THF, SystemOnTPTP and HOL-ATPs

### TPTP THF library

- ▶ TPTP library: <http://www.tptp.org> → Problems
- ▶ THF problems have a '^' in the problem name
- ▶ Example: Cantor's Theorem ([SET557^1.p](#))

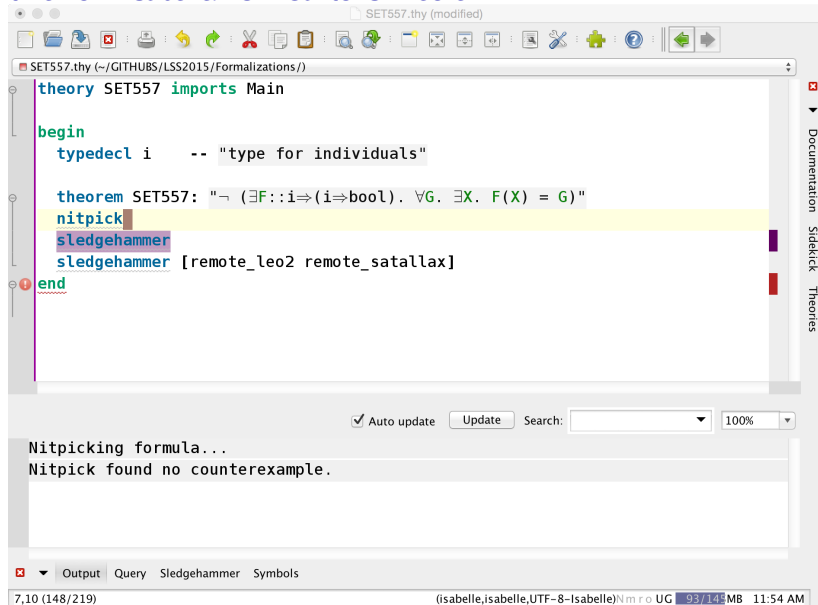
$$\neg \exists F_{t \rightarrow (t \rightarrow o)} \forall G_{t \rightarrow o} \exists X_t. FX = G$$

### SystemOnTPTP and HOL-ATPs

- ▶ SystemOnTPTP: <http://www.tptp.org> → SystemOnTPTP
- ▶ Example:

$$\exists P_{t \rightarrow o} \forall X_t. PX$$

## Short Demo in Isabelle/HOL: Cantor's Theorem



SET557.thy (modified)

SET557.thy (~/GITHUBS/LSS2015/Formalizations/)

```
theory SET557 imports Main
begin
  typedcl i    -- "type for individuals"

  theorem SET557: "¬ (∃F::i⇒(i⇒bool). ∀G. ∃X. F(X) = G)"
  nitpick
  sledgehammer
  sledgehammer [remote_leo2 remote_satallax]
end
```

☒ Auto update    Update    Search:     100%

Nitpicking formula...

Nitpick found no counterexample.

Output   Query   Sledgehammer   Symbols

7,10 (148/219)    (isabelle,isabelle,UTF-8-Isabelle)N m r o UG 93/14 MB 11:54 AM

See demo file SET557.thy

### ATPs as external reasoners in Interactive Proof Assistants

[KaliszykUrban, Learning-Assisted Automated Reasoning with Flyspeck, JAR, 2014]

- ▶ Flyspeck project: formal proof (in HOL-light) of Kepler's Conjecture
- ▶ automation of 14185 theorems studied by Kaliszyk and Urban
- ▶ they developed AI architecture employing various external ATPs in which 39 % of the theorems could be proved in a push-button mode in 30 seconds of real time on a fourteen-CPU workstation
- ▶ subset of 1419 theorems extracted from Flyspeck theorems
- ▶ **next slide:** performance of THF0 provers on these 1419 problems

**Table 7** All ATP re-proving with 30s time limit on 10 % of problems

| Prover   | Theorem (%) | Unique | SOTAC | $\Sigma$ -SOTAC | CounterSat (%) | Processed |
|----------|-------------|--------|-------|-----------------|----------------|-----------|
| Isabelle | 587 (41.3)  | 39     | 0.201 | 118.09          | 0 (0.0)        | 1419      |
| Epar     | 545 (38.4)  | 9      | 0.131 | 71.18           | 0 (0.0)        | 1419      |
| Z3       | 513 (36.1)  | 17     | 0.149 | 76.49           | 0 (0.0)        | 1419      |
| E 1.6    | 463 (32.6)  | 0      | 0.101 | 46.69           | 0 (0.0)        | 1419      |
| LEO2-po1 | 441 (31.0)  | 1      | 0.106 | 46.85           | 0 (0.0)        | 1419      |
| Vampire  | 434 (30.5)  | 3      | 0.107 | 46.44           | 0 (0.0)        | 1419      |
| CVC3     | 411 (28.9)  | 4      | 0.111 | 45.76           | 0 (0.0)        | 1419      |
| Satallax | 383 (26.9)  | 7      | 0.130 | 49.69           | 1 (0.0)        | 1419      |
| Yices    | 360 (25.3)  | 0      | 0.097 | 35.06           | 0 (0.0)        | 1419      |
| iProver  | 348 (24.5)  | 0      | 0.088 | 30.50           | 9 (0.6)        | 1419      |
| Prover9  | 345 (24.3)  | 0      | 0.087 | 30.07           | 0 (0.0)        | 1419      |
| Metis    | 331 (23.3)  | 0      | 0.085 | 28.23           | 0 (0.0)        | 1419      |
| SPASS    | 326 (22.9)  | 0      | 0.081 | 26.46           | 0 (0.0)        | 1419      |
| leanCoP  | 305 (21.4)  | 1      | 0.092 | 27.96           | 0 (0.0)        | 1419      |
| AltErgo  | 281 (19.8)  | 1      | 0.100 | 28.14           | 0 (0.0)        | 1419      |
| LEO2-po2 | 53 (3.7)    | 0      | 0.082 | 4.34            | 0 (0.0)        | 1419      |
| Paradox  | 1 (0.0)     | 0      | 0.059 | 0.06            | 259 (18.2)     | 1419      |
| any      | 712 (50.1)  |        |       |                 | 259 (18.2)     | 1419      |

### Theoretical Philosophy and Metaphysics

[Benzmüller&Woltzenlogel-Paleo, Automating Gödel's Ontological Proof, ECAI, 2014]

- ▶ First-time verification/automation of a modern ontological argument

Gödel's/Scott's proof of the existence of God

- ▶ Remember Leibniz: Two debating philosophers . . . Calculamus!
- ▶ Gödel's argument employs Higher-Order Modal Logic



# Gödel's God in TPTP THF

```
>  
>  
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p
```

Leo-II tries to prove

=====

Goedel's Theorem T3: "Necessarily, God exists"

```
thf(thmT3,conjecture,  
  ( v  
    @ ( mbox  
      @ ( mexists_ind  
        @ ^ [X: mu] :  
          ( g @ X ) ) ) ).
```

Assumptions: D1, C, T2, D3, A5

. searching for proof ..

\*\*\*\*\*

\* Proof found \*

\*\*\*\*\*

% SZS status Theorem for Notwendigerweise-existiert-Gott.p

. generating proof object □

Provers can be called remotely in Miami — no local installation needed!

Download our experiments from

<https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF>



The screenshot shows the Isabelle website homepage. At the top, there's a navigation bar with links like Home-FU, 2012-Watson, Homepage, 2012-FOL, SPIEGEL, 2012-FOL-Home, GMail, Google Maps, M&M, SigmaOnline, Kita, Sigma, Kalender, Beliebt, and Google Maps. Below this is the Isabelle logo (a stack of colored blocks) and the title "Isabelle". To the right are logos for the University of Cambridge Computer Laboratory and the University of Munich. A sidebar on the left contains links: Home, Overview, Installation, Documentation, Community, and Site Mirrors. The main content area has a purple header "What is Isabelle?" followed by a paragraph describing Isabelle as a generic proof assistant. Below this is another purple header "Now available: Isabelle2013" with download buttons for Mac OS X and Linux/Windows. A "Some highlights:" section lists recent improvements. At the bottom, a "Distribution & Support" section provides information about the BSD license and support channels. The footer indicates the last update was on 2013-03-09 12:21:39.

Isabelle

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle is developed at University of Cambridge ([Larry Paulson](#)), Technische Universität München ([Tobias Nipkow](#)) and Université Paris-Sud ([Makarius Wenzel](#)). See the [Isabelle overview](#) for a brief introduction.

Now available: Isabelle2013

Download for Mac OS X

Download for Linux - Download for Windows

Some highlights:

- Improvements of Isabelle/Scala and IsabellejEdit Prover IDE.
- Advanced build tool based on Isabelle/Scala.
- Updated manuals: isar-ref, implementation, system.
- Pure: improved support for block-structured specification contexts.
- HOL tool enhancements: Sledgehammer, Lifting, Quickcheck.
- HOL library enhancements: HOL-Library, HOL-Probability, HOL-Cardinals.
- HOL: New BNF-based (co)datatype package.
- Improved performance thanks to Poly/ML 5.5.0.

See also the cumulative [NEWS](#).

Distribution & Support

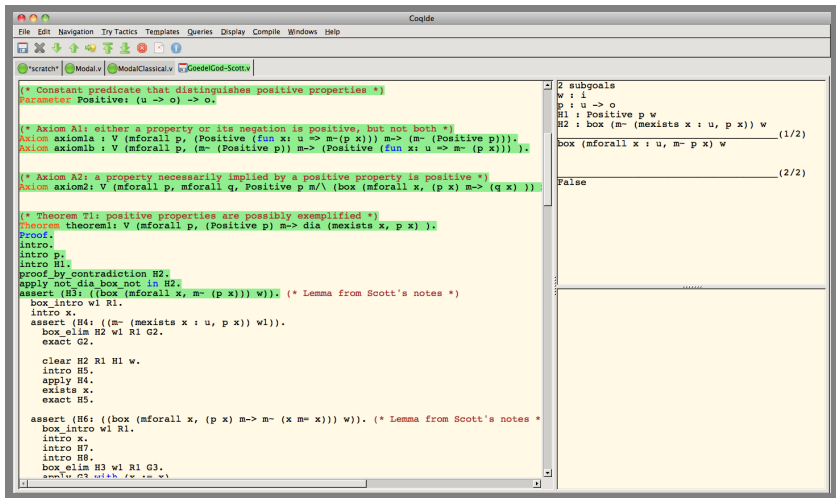
Isabelle is distributed for free under the BSD license. It includes source and binary packages and documentation, see the detailed [installation instructions](#). A vast collection of Isabelle examples and applications is available from the [Archive of Formal Proofs](#). Support is available by ample [documentation](#), the [Isabelle community Wiki](#), and the following mailing lists:

- [isabelle-users@cl.cam.ac.uk](mailto:isabelle-users@cl.cam.ac.uk) provides a forum for Isabelle users to discuss problems, exchange information, and make announcements. Users of official Isabelle releases should [subscribe](#) or see the [archive](#) (also available via [Google groups](#) and [Narkive](#)).
- [isabelle-dev@in.tum.de](mailto:isabelle-dev@in.tum.de) covers the Isabelle development process, including intermediate repository versions, and administrative issues concerning the website or testing infrastructure. Early adopters of [repository versions](#) should [subscribe](#) or see the [archive](#) (also available at [mail-archive.com](mailto:mail-archive.com) or [gmane.org](mailto:gmane.org)).

Last updated: 2013-03-09 12:21:39

See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:  
<http://afp.sourceforge.net/entries/GoedelGod.shtml>

# Gödel's God in Coq



```
(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiom1 : V (mforall p, (Positive (fun x: u => m~(p x))) m-> (m~ (Positive p))).
Axiom axiom1b : V (mforall p, (m~ (Positive p)) m-> (Positive (fun x: u => m~ (p x))) ).

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x) ))).

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x) ).
Proof.
  intro.
  intro p.
  intro R1.
  proof_by_contradiction H2.
  apply not_dia_box_not_in H2.
  assert (H3: ((box (mforall x, m~ (p x))) w)). (* Lemma from Scott's notes *)
  box_intro w1 R1.
  intro x.
  assert (H4: ((m~ (mexists x : u, p x)) w1)).
  box_elim H2 w1 R1 G2.
  exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

  assert (H6: ((box (mforall x, (p x) m-> m~ (x m= x))) w)). (* Lemma from Scott's notes *)
  box_intro w1 R1.
  intro x.
  intro H7.
  intro H8.
  box_elim H3 w1 R1 G3.
  smt_w G3 with (w := w).
```

2 subgoals  
w : 1  
p : u -> o  
H1 : Positive p w  
H2 : box (m~ (mexists x : u, p x)) w  
box (mforall x : u, m~ p x) w (1/2)  
False (2/2)

See verifiable Coq document at:

<https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq>

## Some References (selection; see also the references therein)

- ▶ (Own) Lecture Courses
  - ▶ [Semantics of Higher-Order Logic](#), IT University of Copenhagen, 2007.
  - ▶ [Automated Theorem Proving in Higher-Order Logics](#), TU Darmstadt, 2006.
  - ▶ [Semantics of Higher-Order Logic](#) (w. Chad Brown), ESSLLI, 2006.
- ▶ Lambda Calculus and Lambda Cube
  - ▶ [The Lambda Calculus](#) (J. Alama), Stanford Encyclopedia of Philosophy, 2013.
  - ▶ [Lambda Calculus with Types](#) (H. Barendregt, W. Dekkers, R. Statman), Cambridge University Press, 2013.
  - ▶ [Henk: a typed intermediate language](#) (S.P. Jones, E. Meijer), 1997.
- ▶ Classical Higher-Order Logic
  - ▶ [Church's Type Theory](#) (P. Andrews), The Stanford Encyclopedia of Philosophy, Spring 2014 Edition
  - ▶ [An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof](#) (P. Andrews), Kluwer, 2002.
  - ▶ [The Seven Virtues of Simple Type Theory](#) (W. Farmer), Journal of Applied Logic, 6:267-286.
  - ▶ [Higher-Order Semantics and Extensionality](#) (C. Benz Müller, C. Brown, M. Kohlhase), JSL, 69(4):1027-1088, 2004.
  - ▶ [Intensional models for the theory of types](#) (R. Muskens), JSL 72(1): 98-118, 2007.
  - ▶ [Cut-Simulation and Impredicativity](#) (C. Benz Müller, C. Brown, M. Kohlhase), LMCS, 5(1:6):1-21, 2009.

## Some References (selection; see also the references therein)

- ▶ Automated Theorem Proving in Classical Higher-Order Logic
  - ▶ [Automation of Higher-Order Logic](#) (C. Benz Müller, D. Miller), Handbook of the History of Logic, Volume 9, Elsevier, 2014.
  - ▶ [Higher-Order Automated Theorem Provers](#) (C. Benz Müller), All about Proofs, Proof for All, 2015.
  - ▶ [Automated Reasoning in Higher-Order Logic using the TPTP THF Infrastructure](#) (G. Sutcliffe, C. Benz Müller), Journal of Formalized Reasoning, 3(1):1-27, 2010.
- ▶ Automated Theorem Provers and Model Finders for Classical Higher-Order Logic
  - ▶ [The Higher-Order Prover LEO-II](#) (C. Benz Müller, L. Paulson, N. Sultana, F. Theiß), In Journal of Automated Reasoning, 2015.
  - ▶ [Satallax: An Automatic Higher-Order Prover](#) (C. Brown), IJCAR, 2012.
  - ▶ [Nitpick: A Counterexample Generator for Higher-Order Logic Based on a Relational Model Finder](#) ( J. Blanchette, Tobias Nipkow), ITP, 2010.
- ▶ Isabelle/HOL proof assistant
  - ▶ [Isabelle/HOL - A Proof Assistant for Higher-Order Logic](#) (T. Nipkow, L.C. Paulson, M. Wenzel), Springer, 2004
  - ▶ [Concrete Semantics - With Isabelle/HOL](#) (T. Nipkow, G. Klein), Springer, 2014.

## Higher-Order Modal Logics (Lectures 3)

Christoph Benzmüller<sup>1</sup> and Bruno Woltzenlogel-Paleo



---

<sup>1</sup>Supported by DFG Heisenberg Fellowship BE 2501/9-1/2

$\Box P$

P is necessary, P is obligatory, P is known,  
P is believed, always P ...

$\Diamond P$

P is possible, P is permissible, P is epistemically possible,  
P is doxastically possible, eventually P ...

$\Box$  and  $\Diamond$  are not truth-functional

HOL can be extended by  $\Box P$  and  $\Diamond P$  to obtain HOML

$$\Box P$$

P is necessary, P is obligatory, P is known,  
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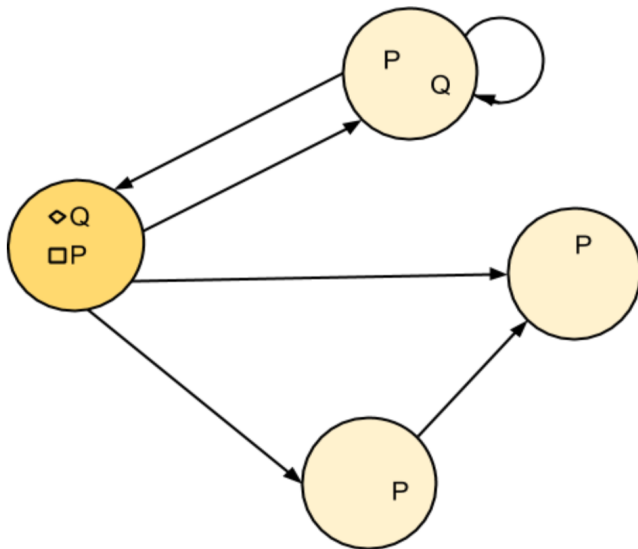
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$\Box$  and  $\Diamond$  are not truth-functional

HOL can be extended by  $\Box P$  and  $\Diamond P$  to obtain HOML

# HOML: Motivation

## Kripke Semantics - Possible Worlds



## HOML: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

HOML Language:

$$\begin{aligned} s, t ::= & P_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ & (\neg_{o \rightarrow o} s_o) \mid ((\vee_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha s_o) \\ & (\Box_{o \rightarrow o} s_o) \end{aligned}$$

constant symbols

variable symbols

lambda abstraction

application

negation

disjunction

universal quantification

modal box operator

Terms of type  $o$ : formulas

Other logical connectives can be defined, e.g.  $\Diamond s$  stands for  $\neg \Box \neg s$

## HOML: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

HOML Language:

$$s, t ::= P_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ (\neg_{o \rightarrow o} s_o) \mid ((\vee_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha s_o)$$

$$(\Box_{o \rightarrow o} s_o)$$

constant symbols

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application

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modal box operator

Terms of type  $o$ : formulas

Other logical connectives can be defined, e.g.  $\Diamond s$  stands for  $\neg \Box \neg s$

## HOML: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

HOML Language:

$$\begin{aligned} s, t ::= & P_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ & (\neg_{o \rightarrow o} s_o) \mid ((\vee_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha s_o) \\ & (\Box_{o \rightarrow o} s_o) \end{aligned}$$

constant symbols

variable symbols

lambda abstraction

application

negation

disjunction

universal quantification

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## $\alpha$ -conversion

... as before ...

## Substitution

... as before ...

## $\beta$ -reduction and $\eta$ -reduction

... as before ...



## Semantics of HOML investigated in

- ▶ [D. Gallin, Intensional and Higher-Order Modal Logic, North Holland, 1975]
- ▶ [R. Muskens, Higher Order Modal Logic, Handbook of Modal Logic, 2006]
- ▶ [Benzmüller and Woltzenlogel Paleo, Automating Gödel's Ontological Proof . . . , ECAI, 2014]
- ▶ our interest: combination of Kripke style models and Henkin semantics

### A Frame $D$

... as before ...

### A Model $M$

for HOML is a quadruple  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$ , where

- ▶  $W$  is a set of worlds (or states);
- ▶  $R$  is an accessibility relation between the worlds in  $W$ ;
- ▶  $D$  is a frame;
- ▶ for each  $w \in W$ ,  $\{I_w\}_{w \in W}$  is a family of typed interpretation functions mapping constant symbols  $p_\alpha$  to appropriate elements of  $D_\alpha$ , called the denotation of  $p_\alpha$  in world  $w$ ;
- ▶ the logical connectives  $\neg$ ,  $\vee$ ,  $\forall$ , and  $\Box$  are always given the standard denotations;
- ▶ moreover, it is assumed that the domains  $D_{\alpha \rightarrow \alpha \rightarrow o}$  contain the respective identity relations on objects of type  $\alpha$ .

## Variable Assignment

... as before ...

## Interpretation/Value of a HOML term

The value  $\|s_\alpha\|^{M,g,w}$  of a HOML term  $s_\alpha$  on a model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  in a world  $w \in W$  under variable assignment  $g$  is an element  $d \in D_\alpha$  defined in the following way:

1.  $\|P_\alpha\|^{M,g,w} = I_w(P_\alpha)$
2.  $\|x_\alpha\|^{M,g,w} = g(x_\alpha)$
3.  $\|(s_{\alpha \rightarrow \beta} t_\alpha)_\beta\|^{M,g,w} = \|s_{\alpha \rightarrow \beta}\|^{M,g,w}(\|t_\alpha\|^{M,g,w})$
4.  $\|(\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta}\|^{M,g,w} = \text{the function } f \text{ from } D_\alpha \text{ to } D_\beta \text{ such that } f(d) = \|s_\beta\|^{M,g[d/x_\alpha],w} \text{ for all } d \in D_\alpha$
5.  $\|(\neg_{o \rightarrow o} s_o)_o\|^{M,g,w} = T \text{ iff } \|s_o\|^{M,g,w} = F$
6.  $\|((\vee_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g,w} = T \text{ iff } \|s_o\|^{M,g,w} = T \text{ or } \|t_o\|^{M,g,w} = T$
7.  $\|(\forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha s_o))_o\|^{M,g,w} = T \text{ iff for all } d \in D_\alpha \text{ we have } \|s_o\|^{M,g[d/x_\alpha],w} = T$
8.  $\|(\Box_{o \rightarrow o} s_o)_o\|^{M,g,w} = T \text{ iff for all } v \in W \text{ with } wRv \text{ we have } \|s_o\|^{M,g,v} = T$

## Variable Assignment

... as before ...

## Interpretation/Value of a HOML term

The value  $\|s_\alpha\|^{M,g,w}$  of a HOML term  $s_\alpha$  on a model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  in a world  $w \in W$  under variable assignment  $g$  is an element  $d \in D_\alpha$  defined in the following way:

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5.  $I(\neg_{o \rightarrow o}) = \text{not} \in D_{o \rightarrow o}$  such that:  $\text{not}(F) = T$  and  $\text{not}(T) = F$
6.  $I(\vee_{o \rightarrow o \rightarrow o}) = \text{or} \in D_{o \rightarrow o \rightarrow o}$  such that:  $\text{or}(a, b) = T$  iff  $a = T$  or  $b = T$
7.  $I(\forall_{(\alpha \rightarrow o) \rightarrow o}) = \text{All} \in D_{(\alpha \rightarrow o) \rightarrow o}$  such that: for  $p \in D_{\alpha \rightarrow o}$  we have  $\text{All}(p) = T$  iff  $p(d) = T$  for all  $d \in D_\alpha$
8.  $\|(\Box_{o \rightarrow o} s_o)_o\|^{M,g,w} = T$  **iff for all**  $v \in W$  **with**  $wRv$  **we have**  $\|s_o\|^{M,g,v} = T$

## Another form of presenting the semantics of HOML

**HOML**       $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma \varphi \mid \exists x_\gamma \varphi$

► Possible world semantics via satisfaction/forcing relation

$M, g, s \Vdash \neg\varphi$       iff    not  $M, g, s \Vdash \varphi$

$M, g, s \Vdash \varphi \wedge \psi$     iff     $M, g, s \Vdash \varphi$  and  $M, g, s \Vdash \psi$

...

$M, g, s \Vdash \Box\varphi$       iff     $M, g, u \Vdash \varphi$  for all  $u$  with  $r(s, u)$

...

$M, g, s \Vdash \forall x_\gamma \varphi$     iff     $M, [d/x]g, s \Vdash \varphi$  for all  $d \in D_\gamma$

...

## Truth in a Model / Validity

A formula  $s_o$  is true in model  $M$  for world  $w$  under assignment  $g$  if and only if  $\|s_o\|^{M,g,w} = T$ ; this is also denoted as  $M, g, w \models s_o$ .

A formula  $s_o$  is called valid in  $M$  if and only if  $M, g, w \models s_o$  for all  $w \in W$  and all assignments  $g$ ; this is also denoted as  $M \models s_o$ .

A formula  $s_o$  is called valid, which we denote by  $\models s_o$ , if and only if  $M \models s_o$  for all  $M$ .

Finally, we define  $\models \varphi$ , where  $\varphi$  is a set of HOML formulas, if and only if  $\models s$  for all  $s \in \varphi$ .

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## HOML: Local and Global Consequence

### Local Consequence

Let  $\varphi$  and  $\psi$  be set of HOML formulas. We define

$$\varphi \models_l \psi \quad (\text{local consequence})$$

if and only if for each model  $M$  and world  $w$  we have:

$$M, w \models \varphi \text{ implies } M, w \models \psi$$

$$\forall w (M, w \models \varphi \text{ implies } M, w \models \psi)$$

### Global Consequence

Let  $\varphi$  and  $\psi$  be set of HOML formulas. We define

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Note the difference in the scope of quantification over worlds!

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Note the difference in the scope of quantification over worlds!

### Standard and Henkin Models (as before)

In a standard model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  we have

- ▶  $D_{\alpha \rightarrow \beta} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )

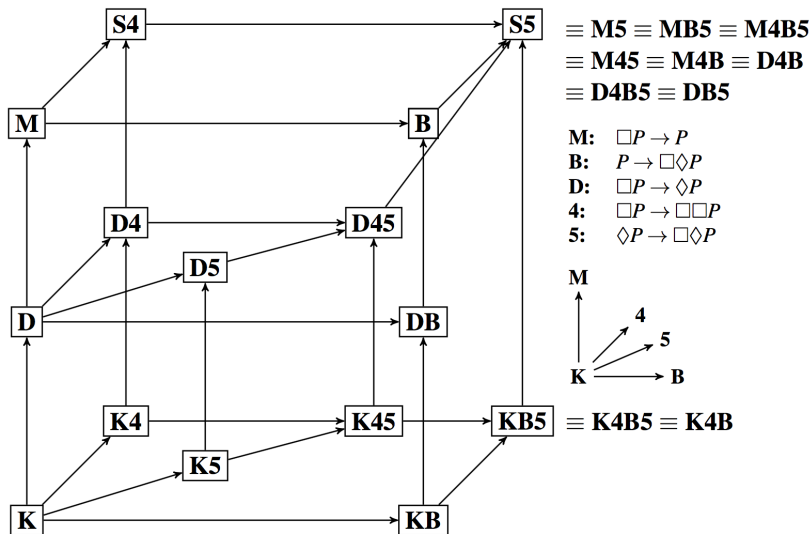
In a Henkin model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  we only require

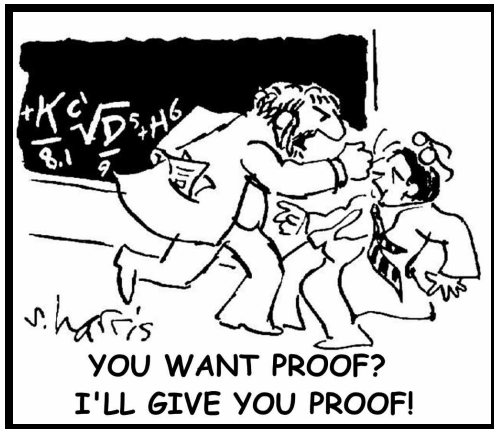
- ▶  $D_{\alpha \rightarrow \beta} \subseteq \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )
- ▶ the valuation function  $\| \cdot \|^{M, g, w}$  from above is total (every term denotes)

Any standard model is obviously also a Henkin model.

We consider Henkin models in the remainder.

# HOML: The Modal Logic Cube





**How to automate HOML?**

## Embedding HOML in HOL

Challenge: No provers for Higher-order Modal Logic (HOML)

Our solution: Embedding in **Church's Simple Type Theory**  
**Higher-order Classical Logic** (HOL)  
Then use existing HOL theorem provers for reasoning in HOML

[BenzmüllerPaulson, Logica Universalis, 2013]

Assumption: Henkin semantics for both HOML and HOL

Previous empirical findings:

Embedding of First-order Modal Logic in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

## Remember: Standard Translation to FOL for Propositional ML

### Standard Translation

$ST_x(p) = P(x)$  (p atomic formula;  $P(x)$  is true when  $p$  holds for world  $x$ )

$$ST_x(\top) = \top$$

$$ST_x(\perp) = \perp$$

$$ST_x(\neg\varphi) = \neg ST_x(\varphi)$$

$$ST_x(\varphi \vee \psi) = ST_x(\varphi) \vee ST_x(\psi)$$

$$ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi)$$

$$ST_x(\varphi \rightarrow \psi) = ST_x(\varphi) \rightarrow ST_x(\psi)$$

$$ST_x(\Diamond_m \varphi) = \exists y(R_m(x, y) \wedge ST_y(\varphi))$$

$$ST_x(\Box_m \varphi) = \forall y(R_m(x, y) \rightarrow ST_y(\varphi))$$

### Standard Translation Example

$\Box \Diamond p$  translates to  $\forall y(R(x, y) \rightarrow (\exists z(R(y, z) \wedge P(z))))$

### Standard Translation Initialisation

$\varphi$  translates to  $\forall x ST_x(\varphi)$



## Embedding HOML in HOL

**HOML**       $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\mu \varphi \mid \exists x_\mu \varphi$

**HOL**       $s, t ::= C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha t_o$

**HOML in HOL:**    **HOML** formulas  $\varphi$  are mapped to **HOL** predicates  $\varphi_{\iota \rightarrow o}$   
 $\mu$  is an additional base type for individuals

|               |     |                                                                                                                      |
|---------------|-----|----------------------------------------------------------------------------------------------------------------------|
| $\neg$        | $=$ | $\lambda\varphi_{\iota \rightarrow o} \lambda w_\iota \neg\varphi w$                                                 |
| $\wedge$      | $=$ | $\lambda\varphi_{\iota \rightarrow o} \lambda\psi_{\iota \rightarrow o} \lambda w_\iota (\varphi w \wedge \psi w)$   |
| $\rightarrow$ | $=$ | $\lambda\varphi_{\iota \rightarrow o} \lambda\psi_{\iota \rightarrow o} \lambda w_\iota (\neg\varphi w \vee \psi w)$ |
| $\forall$     | $=$ | $\lambda h_{\mu \rightarrow (\iota \rightarrow o)} \lambda w_\iota \forall d_\mu h d w$                              |
| $\exists$     | $=$ | $\lambda h_{\mu \rightarrow (\iota \rightarrow o)} \lambda w_\iota \exists d_\mu h d w$                              |
| $\Box$        | $=$ | $\lambda\varphi_{\iota \rightarrow o} \lambda w_\iota \forall u_\iota (\neg R w u \vee \varphi u)$                   |
| $\Diamond$    | $=$ | $\lambda\varphi_{\iota \rightarrow o} \lambda w_\iota \exists u_\iota (R w u \wedge \varphi u)$                      |
| <b>valid</b>  | $=$ | $\lambda\varphi_{\iota \rightarrow o} \forall w_\iota \varphi w$                                                     |

**Ax** (polymorphic over  $\mu$ )

The equations in **Ax** are given as axioms to the **HOL** provers!

## Embedding HOML in HOL

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## Embedding HOML in HOL

### Example

**HOML** formula

$\Diamond \exists x G(x)$

**HOML** formula in **HOL**

$\text{valid } (\Diamond \exists x G(x))_{\iota \rightarrow o}$

expansion

$(\lambda \varphi \forall w_{\iota} \varphi w) (\Diamond \exists x G(x))_{\iota \rightarrow o}$

$\beta\eta$ -normalisation

$\forall w_{\iota} ((\Diamond \exists x G(x))_{\iota \rightarrow o} w)$

expansion

$\forall w_{\iota} (((\lambda \varphi_{\iota \rightarrow o} \lambda w_{\iota} \exists u_{\iota} (Rwu \wedge \varphi u)) \exists x G(x))_{\iota \rightarrow o} w)$

$\beta\eta$ -normalisation

$\forall w_{\iota} \exists u_{\iota} (Rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$

syntactic sugar

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$\forall w_{\iota} \exists u_{\iota} (Rwu \wedge ((\lambda h_{\mu \rightarrow (\iota \rightarrow o)} \lambda w_{\iota} \exists d_{\mu} h d w) (\lambda x G(x)))_{\iota \rightarrow o} u)$

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Expansion: user or prover may flexibly choose expansion depth

### What are we doing?

In order to prove that  $\varphi$  is valid in **HOML**,

$\rightarrow$  we instead prove that  $\text{valid } \varphi_{\iota \rightarrow o}$  can be derived from **Ax** in **HOL**.

This can be done with interactive or automated **HOL** theorem provers.

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HOML formula in HOL

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$$\forall w_{\iota} \exists u_{\iota} (Rwu \wedge \exists x G x u)$$

Expansion: user or prover may flexibly choose expansion depth

### What are we doing?

In order to prove that  $\varphi$  is valid in HOML,

$\rightarrow$  we instead prove that  $\text{valid } \varphi_{\iota \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

## Embedding HOML in HOL

### Example

HOML formula

$$\Diamond \exists x G(x)$$

HOML formula in HOL

$$\text{valid } (\Diamond \exists x G(x))_{\iota \rightarrow o}$$

expansion

$$(\lambda \varphi \forall w_{\iota} \varphi w) (\Diamond \exists x G(x))_{\iota \rightarrow o}$$

$\beta\eta$ -normalisation

$$\forall w_{\iota} ((\Diamond \exists x G(x))_{\iota \rightarrow o} w)$$

expansion

$$\forall w_{\iota} (((\lambda \varphi_{\iota \rightarrow o} \lambda w_{\iota} \exists u_{\iota} (Rwu \wedge \varphi u)) \exists x G(x))_{\iota \rightarrow o} w)$$

$\beta\eta$ -normalisation

$$\forall w_{\iota} \exists u_{\iota} (Rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$$

syntactic sugar

$$\forall w_{\iota} \exists u_{\iota} (Rwu \wedge (\exists (\lambda x G(x)))_{\iota \rightarrow o} u)$$

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## Embedding HOML in HOL

### Example

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**HOML** formula in **HOL**

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In order to prove that  $\varphi$  is valid in **HOML**,

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This can be done with interactive or automated **HOL** theorem provers.

## Embedding HOML in HOL: Theoretical Results

### Soundness and Completeness (and Cut-elimination)

$$\models_{\text{Henkin}}^{\text{HOML}} s_o \quad \text{iff} \quad \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{L \rightarrow o} \quad (\text{iff} \quad \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{L \rightarrow o})$$

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$$\models_{\text{Henkin}}^{\text{HOML}} s_o \quad \text{iff} \quad \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\ell \rightarrow o} \quad (\text{iff} \quad \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{\ell \rightarrow o})$$

**Proof sketch (adapts [Benzmüller and Paulson, Logica Universalis, 2013]):**

By contraposition it is sufficient to show

$$\not\models_{\text{Henkin}}^{\text{HOML}} s_o \quad \text{iff} \quad \text{Ax} \not\models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\ell \rightarrow o}$$

One easily gets the proof by choosing the obvious correspondences between  $D$  and  $\mathcal{D}$ ,  $W$  and  $\mathcal{D}_\ell$ ,  $I$  and  $\mathcal{I}$ ,  $g$  and  $\mathcal{g}$ ,  $R$  and  $r_{\ell \rightarrow \ell \rightarrow o}$ , and  $w$  and  $\mathcal{w}$ .  $\square$

Remark: So far we have addressed only base logic K — more soon!



# Embedding of Other Logics in HOL: Theoretical Results

## Soundness and Completeness (and Cut-elimination)

$$\models^L s_o \quad \text{iff} \quad \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{L \rightarrow o} \quad (\text{iff} \quad \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{L \rightarrow o})$$

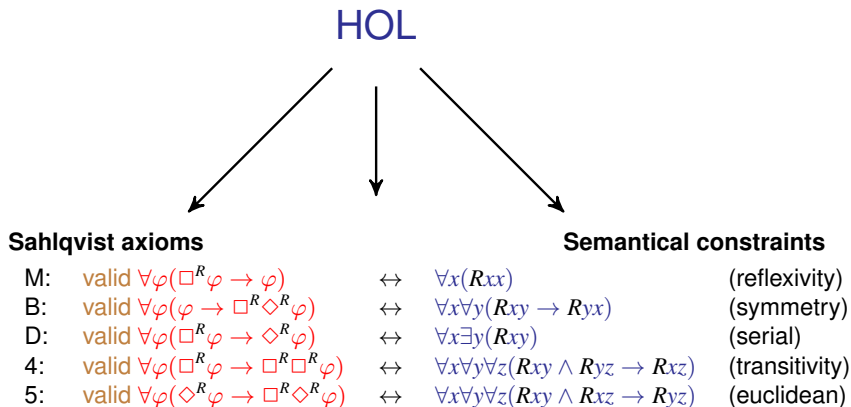
Logic L:

- ▶ Higher-order Modal Logics
- ▶ First-order Multimodal Logics
- ▶ Propositional Multimodal Logics
- ▶ Quantified Conditional Logics
- ▶ Propositional Conditional Logics
- ▶ Intuitionistic Logics
- ▶ Access Control Logics
- ▶ Logic Combinations
- ▶ ... more is on the way ... including:
  - ▶ Description Logics
  - ▶ Nominal Logics
  - ▶ Multivalued Logics (SIXTEEN)
  - ▶ Logics based on Neighborhood Semantics
  - ▶ (Mathematical) Fuzzy Logics
  - ▶ Paraconsistent Logics



## Flexibility in HOML

## Postulating modal axioms or semantical constraints



## Embedding HOML in HOL: Logics beyond K

$$(\Diamond \exists x Pfx \wedge \Box \forall y (\Diamond Py \rightarrow Qy)) \rightarrow \Diamond \exists z Qz$$

$$\text{valid } (\Diamond \exists x Pfx \wedge \Box \forall y (\Diamond Py \rightarrow Qy)) \rightarrow \Diamond \exists z Qz$$

## Embedding HOML in HOL: Logics beyond K

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$$\Box = \lambda p \lambda w \forall v (\neg (Rwv) \vee (pv))$$

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$$(\Diamond \exists x Pfx \wedge \Box \forall y (\Diamond Py \rightarrow Qy)) \rightarrow \Diamond \exists z Qz$$

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...

$$\forall w (\neg \neg (\neg \neg \forall v (\neg R w v \vee \neg \neg \forall x \neg P(fx) v) \vee \neg \forall v (\neg R w v \vee \forall y (\neg \neg \forall u (\neg R v u \vee \neg P y u) \vee Q y v))) \vee \neg \forall v (\neg R w v \vee \neg \neg \forall z \neg Q z v))$$



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### Axiomatization of properties of accessibility relation $R$

Logic K: no axioms

Logic T: (*reflexive R*) — which expands into  $\forall x Rxx$

Logic S5: (*reflexive R*)  $\wedge$  (*symmetric R*)  $\wedge$  (*transitive R*)

Logic ... ..

## Embedding HOML in HOL: Logics beyond K

$$(\Diamond \exists x Pfx \wedge \Box \forall y (\Diamond Py \rightarrow Qy)) \rightarrow \Diamond \exists z Qz$$

*valid*  $(\Diamond \exists x Pfx \wedge \Box \forall y (\Diamond Py \rightarrow Qy)) \rightarrow \Diamond \exists z Qz$

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...

$$\forall w (\neg \neg (\neg \neg \forall v (\neg R w v \vee \neg \neg \forall x \neg P(fx) v) \vee \neg \forall v (\neg R w v \vee \forall y (\neg \neg \forall u (\neg R v u \vee \neg P y u) \vee Q y v))) \vee \neg \forall v (\neg R w v \vee \neg \neg \forall z \neg Q z v))$$

### Axiomatization of properties of accessibility relation $R$

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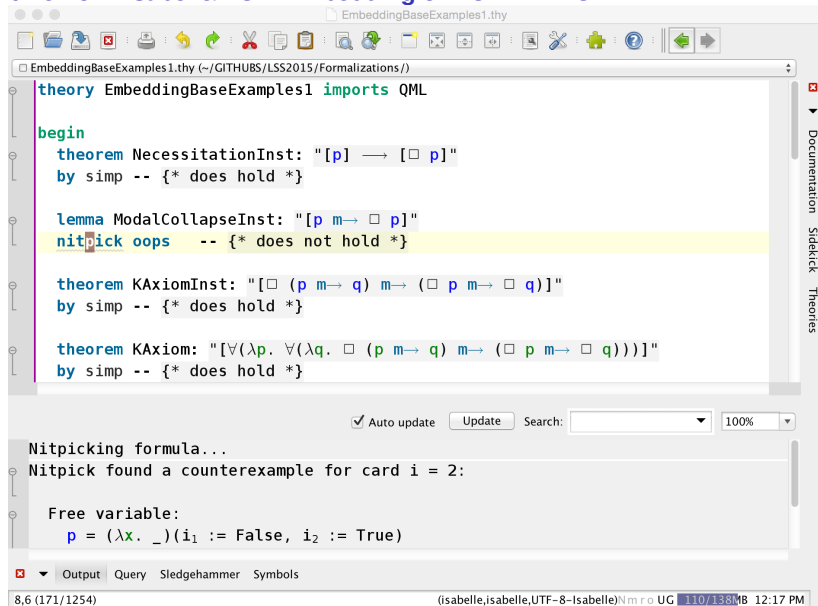
Logic T: (*reflexive*  $R$ ) — which expands into  $\forall x Rxx$

Logic S5: (*reflexive*  $R$ )  $\wedge$  (*symmetric*  $R$ )  $\wedge$  (*transitive*  $R$ )

Logic ...

This automates **HOML** (here **FOML**) with constant domain semantics in **HOL**

# Short Demo in Isabelle/HOL: Embedding of HOML in HOL



EmbeddingBaseExamples1.thy

```
theory EmbeddingBaseExamples1 imports QML
begin
theorem NecessitationInst: "[p]  $\longrightarrow$  [ $\Box$  p]"
by simp -- {* does hold *}

lemma ModalCollapseInst: "[p  $\mathbf{m} \longrightarrow$   $\Box$  p]"
nitpickoops -- {* does not hold *}

theorem KAXiomInst: "[ $\Box$  (p  $\mathbf{m} \longrightarrow$  q)  $\mathbf{m} \longrightarrow$  ( $\Box$  p  $\mathbf{m} \longrightarrow$   $\Box$  q)]"
by simp -- {* does hold *}

theorem KAXiom: "[ $\forall (\lambda p. \forall (\lambda q. \Box (p \mathbf{m} \longrightarrow q) \mathbf{m} \longrightarrow (\Box p \mathbf{m} \longrightarrow \Box q)))$ ]"
by simp -- {* does hold *}
```

☒ Auto update    Update    Search:    100%

Nitpicking formula...

Nitpick found a counterexample for card i = 2:

Free variable:

p = ( $\lambda x. \_$ )(i<sub>1</sub> := False, i<sub>2</sub> := True)

Output    Query    Sledgehammer    Symbols

8,6 (171/1254)    (isabelle,isabelle,UTF-8-Isabelle)Nm r o UG 110/138MB 12:17 PM

See demo file EmbeddingBaseExamples1.thy

## Embedding HOML in HOL: Multimodal Logics

Instead of

$$\Box = \lambda\varphi_{l \rightarrow o} \lambda w_l \forall u_l (\neg R w u \vee \varphi u)$$

consider

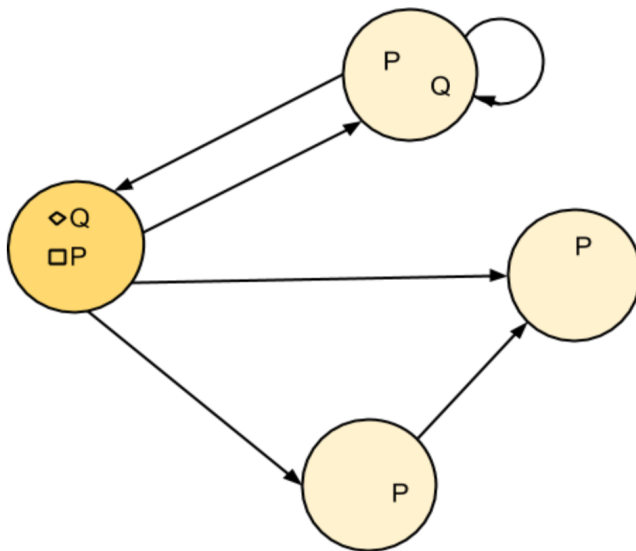
$$\Box = \lambda r_{l \rightarrow l \rightarrow o} \lambda\varphi_{l \rightarrow o} \lambda w_l \forall u_l (\neg r w u \vee \varphi u)$$

Multiple box operators:  $\Box_{knowledgeBen}, \Box_{knowledgePeter}, \Box_{commonKnowledge}, \dots$

Bridge rules (quantification over propositional variables)

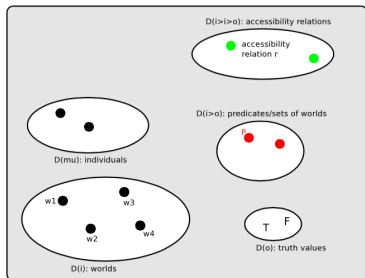
$$\forall p (\Box_{commonKnowledge} p \rightarrow \Box_{knowledgeBen} p)$$

## Embedding HOML in HOL: Possibilist vs. Actualist Quantification



# Embedding HOML in HOL: Possibilist vs. Actualist Quantification

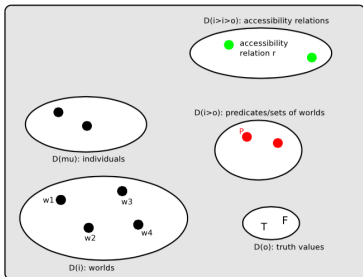
## Constant Domain



$$\Pi = \lambda h \lambda w_t \forall x_\mu h x w$$

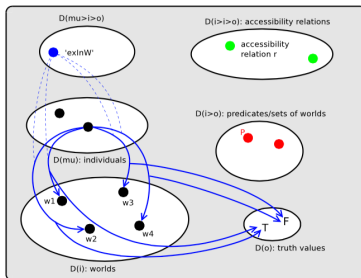
# Embedding HOML in HOL: Possibilist vs. Actualist Quantification

## Constant Domain



$$\Box = \lambda h \lambda w_\iota \forall x_\mu h x w$$

## Varying and Cumulative Domain

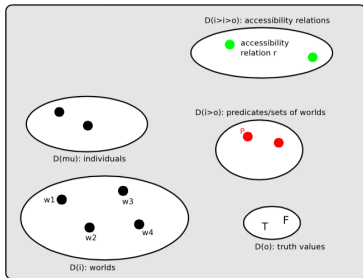


$$\Box^{va} = \lambda h \lambda w_\iota \forall x_\mu (\neg \text{exInW}_{xw} \vee h x w)$$

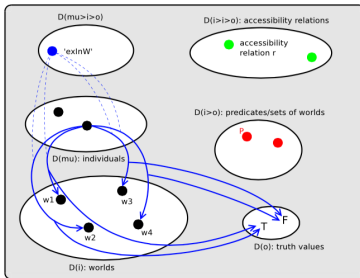


# Embedding HOML in HOL: Possibilist vs. Actualist Quantification

## Constant Domain



## Varying and Cumulative Domain



$$\Pi = \lambda h \lambda w_l \forall x_\mu h x_\mu w$$

domains are non-empty

$$\Pi^{va} = \lambda h \lambda w_l \forall x_\mu (\neg \text{exInW}_{xw} \vee h x_\mu w)$$

denotation (constants & functions)

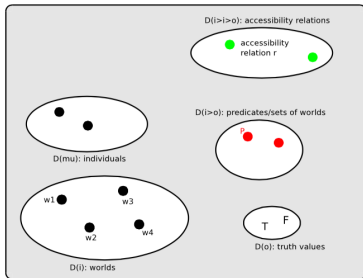
$$\forall w_l \exists x_\mu \text{exInW}_{xw}$$

$$\forall w_l \text{exInW}_{cw}$$

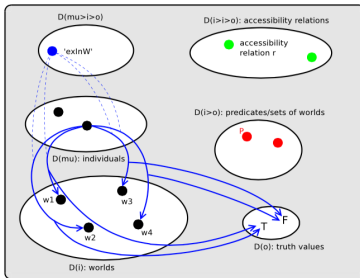
$$\forall w_l (\text{exInW}_{t^1 w} \wedge \dots \wedge \text{exInW}_{t^n w} \supset \text{exInW}_{(f t^1 \dots t^n) w})$$

# Embedding HOML in HOL: Possibilist vs. Actualist Quantification

## Constant Domain



## Varying and Cumulative Domain



$$\Pi = \lambda h \lambda w_l \forall x_\mu h x w$$

$$\Pi^{va} = \lambda h \lambda w_l \forall x_\mu (\neg \text{exInW}_{xw} \vee h x w)$$

domains are non-empty

$$\forall w_l \exists x_\mu \text{exInW}_{xw}$$

denotation (constants & functions)

$$\forall w_l \text{exInW}_{cw}$$

$$\forall w_l (\text{exInW}_{t^1 w} \wedge \dots \wedge \text{exInW}_{t^n w} \supset \text{exInW}_{(f t^1 \dots t^n) w})$$

cumulative domains

$$\forall x, v, w (\text{exInW}_{xv} \wedge r v w \supset \text{exInW}_{xw})$$

## Embedding HOML in HOL: Possibilist vs. Actualist Quantification

$$\forall = \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_{\iota} \forall d_{\gamma} h d w \quad (\text{possibilist / constant dom.})$$

becomes

$$\forall^{va} = \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_{\iota} \forall d_{\gamma} (\mathbf{ExInW} d w \rightarrow h d w) \quad (\text{actualist / varying dom.})$$

where **ExInW** is an existence predicate.

Additional axioms (optional):

- ▶ domains are non-empty  $\forall w_{\iota} \exists x_{\mu} \mathbf{exInW} x w$
- ▶ denotation (constants & functions)  $\forall w_{\iota} \mathbf{exInW} c w$   
 $\forall w_{\iota} (\mathbf{exInW} t^1 w \wedge \dots \wedge \mathbf{exInW} t^n w \supset \mathbf{exInW} (f t^1 \dots t^n) w)$

Cumulative domains:  $\forall x \forall v \forall w (\mathbf{exInW} x v \wedge r v w \supset \mathbf{exInW} x w)$

## Short Demo in Isabelle/HOL on Barcan Formulas?

```
theory BarcanFormulas imports QML
begin
-- {*
We check whether (an instance of) the Barcan formula schema is true:
"If there could have been an object that met a given condition,
then there is an object that could have met the condition."
*}
theorem BF: "[ $\Diamond(\exists(\lambda x. P\ x)) \multimap \exists(\lambda x. \Diamond(P\ x))$ ]"
by auto

-- {*
We check whether (an instance of) the converse Barcan formula schema is true:
"If there is an object that could have met the condition,
then there could have been an object that met the condition."
*}
theorem CBF: "[ $\exists(\lambda x. \Diamond(P\ x)) \multimap \Diamond(\exists(\lambda x. P\ x))$ ]"
by auto
```

☒ Auto update    Update    Search:     100%

```
theorem BF: [ $\lambda w. \Diamond(\exists ?P) w \longrightarrow (\exists x. \Diamond(?P\ x)\ w)$ ]
```

☒ ▾    Output    Query    Sledgehammer    Symbols

9,7 (310/777)    (isabelle,isabelle,UTF-8-Isabelle)Nm r o UG 123/142MB 12:37 PM

See demo files `BarcanFormulas.thy` and `BarcanFormulasVar.thy`

# Embedding HOML in HOL: TPTP THF

```
1  %----The base type $i (already built-in) stands here for worlds and
2  %----mu for individuals; $o (also built-in) is the type of Booleans
3  thf(mu_type,type,(mu:$tType)).
4  %----Reserved constant r for accessibility relation
5  thf(r,type,(r:$i>$i>$o)).
6  %----Modal logic operators not, or, and, implies, box, diamond
7  thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
8  thf(mnot,definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))).
9  thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
10 thf(mor,definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)|(Psi@W))))).
11 thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
12 thf(mand,definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
13 thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
14 thf(mimplies,definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
15 thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
16 thf(mbox,definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V)|(A@V))))).
17 thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
18 thf(mdia,definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V)&(A@V))))).
19 %----Quantifiers (constant domains) for individuals and propositions
20 thf(mforall_ind_type,type,(mforall_ind:(mu>$i>$o)>$i>$o)).
21 thf(mforall_ind,definition,(mforall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
22 thf(mforall_indset_type,type,(mforall_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
23 thf(mforall_indset,definition,(mforall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(A@X@W)))).
24 thf(mexists_ind_type,type,(mexists_ind:(mu>$i>$o)>$i>$o)).
25 thf(mexists_ind,definition,(mexists_ind = (^[A:mu>$i>$o,W:$i]:?[X:mu]:(A@X@W)))).
26 thf(mexists_indset_type,type,(mexists_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
27 thf(mexists_indset,definition,(mexists_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:?[X:mu>$i>$o]:(A@X@W)))).
28 %----Definition of validity (grounding of lifted modal formulas)
29 thf(v_type,type,(v:($i>$o)>$o)).
30 thf(mvalid,definition,(v = (^[A:$i>$o]:![W:$i]:(A@W)))).
31 %----Properties of accessibility relations: symmetry
32 thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
33 thf(msymmetric,definition,(msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T)=>(R@T@S))))).
34 %----Here we work with logic KB, i.e., we postulate symmetry for r
35 thf(sym,axiom,(msymmetric@r)).
```

Reading on THF0 syntax: [SutcliffeBenzmüller, J.Formalized Reasoning, 2010]

# Proof Automation with LEO-II

```
>  
>  
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p
```

Leo-II tries to prove

=====

Goedel's Theorem T3: "Necessarily, God exists"

```
thf(thmT3,conjecture,  
  ( v  
    @ ( mbox  
      @ ( mexists_ind  
        @ ^ [X: mu] :  
          ( g @ X ) ) ) ).
```

Assumptions: D1, C, T2, D3, A5

. searching for proof ..

\*\*\*\*\*

\* Proof found \*

\*\*\*\*\*

% SZS status Theorem for Notwendigerweise-existiert-Gott.p

. generating proof object □

Provers can be called remotely in Miami — no local installation needed!

Download our experiments from

https:

[//github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF](https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF)

## Evaluation — How good is the embedding approach?

- ▶ There are no other provers for HOML
- ▶ There are some provers for first-order modal logic (FML)
- ▶ Comparative evaluation done in 2014 for the proof problems in the QMLTP (v1.1) library [Otten and Raths, <http://www.iltp.de/qmltp/>]
- ▶ This library contains 580 FML problems (in fact,  $5 \times 3 \times 580 = 8700$  problems).
- ▶ Metaprover HOL-P: sequentially schedules LEO-II—1.6.2, Satallax—2.7, Isabelle—2013, Nitrox—2013, agsyHOL—1.0
- ▶ Timeout for each HOL prover 120sec of CPU time (HOL-P: 600sec)
- ▶ Timeout for competitor systems: 600sec

## Evaluation — How good is the embedding approach?

FO Modal Logic example:  $(\Diamond \exists x Pfx \wedge \Box \forall y (\Diamond Py \Rightarrow Qy)) \Rightarrow \Diamond \exists z Qz$

```
%> ./FMLtoHOL-P example.thf -timeout 20 -logic s4 -domain varying
```

**Calling HOL Resoners remotely in Miami ... thanks to Geoff Sutcliffe**

- LEO-II says **Theorem** — CPU 0.08s
- Satallax says **Theorem** — CPU 0.03s
- Isabelle says Unknown — CPU 11.93s
- Nitpick says Unknown — CPU 10.62s
- agsyHOL says **Theorem** — CPU 0.55s

```
%> ./FMLtHOL-P example.thf -timeout 20 -logic k -domain constant
```

**Calling HOL Resoners remotely in Miami ... thanks to Geoff Sutcliffe**

- LEO-II says Unknown — CPU 11.93s
- Satallax says **CounterSatisfiable** — CPU 0.04s
- Isabelle says Unknown — CPU 16.19s
- Nitpick says **CounterSatisfiable** — CPU 8.19s
- agsyHOL says Unknown — CPU 10.82s



## Evaluation — FML's (D — constant/varying/cumulative)

No. of solved problems in the QMLTP library

|                                    | <b>MleanSeP</b><br>labelled<br>sequents | <b>MleanTAP</b><br>labelled<br>tableaux | <b>f2p-MSPASS</b><br>instant. &<br>transform. | <b>MleanCoP</b><br>labelled<br>connections | <b>HOL-P</b> |
|------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------------|--------------------------------------------|--------------|
| <b>Logic D, constant domains</b>   |                                         |                                         |                                               |                                            |              |
| Theorem                            | 135                                     | 134                                     | 76                                            | <b>217</b>                                 | 208          |
| Non-Theorem                        | 1                                       | 4                                       | 107                                           | 209                                        | <b>250</b>   |
| Solved                             | 136                                     | 138                                     | 183                                           | 426                                        | <b>458</b>   |
| <b>Logic D, cumulative domains</b> |                                         |                                         |                                               |                                            |              |
| Theorem                            | 130                                     | 120                                     | 79                                            | <b>200</b>                                 | 184          |
| Non-Theorem                        | 4                                       | 4                                       | 108                                           | 224                                        | <b>269</b>   |
| Solved                             | 134                                     | 124                                     | 187                                           | 424                                        | <b>453</b>   |
| <b>Logic D, varying domains</b>    |                                         |                                         |                                               |                                            |              |
| Theorem                            | -                                       | 100                                     | -                                             | <b>170</b>                                 | 163          |
| Non-Theorem                        | -                                       | 4                                       | -                                             | 243                                        | <b>295</b>   |
| Solved                             | -                                       | 104                                     | -                                             | 413                                        | <b>458</b>   |

## Evaluation — FML's (S4 — constant/varying/cumulative)

No. of solved problems in the QMLTP library

|                                     | <b>MleanSeP</b><br>labelled<br>sequents | <b>MleanTAP</b><br>labelled<br>tableaux | <b>f2p-MSPASS</b><br>instant. &<br>transform. | <b>MleanCoP</b><br>labelled<br>connections | <b>HOL-P</b> |
|-------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------------|--------------------------------------------|--------------|
| <b>Logic S4, constant domains</b>   |                                         |                                         |                                               |                                            |              |
| Theorem                             | 197                                     | 220                                     | 111                                           | <b>352</b>                                 | 300          |
| Non-Theorem                         | 1                                       | 4                                       | 36                                            | 82                                         | <b>132</b>   |
| Solved                              | 198                                     | 224                                     | 147                                           | <b>434</b>                                 | 432          |
| <b>Logic S4, cumulative domains</b> |                                         |                                         |                                               |                                            |              |
| Theorem                             | 197                                     | 205                                     | 121                                           | <b>338</b>                                 | 278          |
| Non-Theorem                         | 4                                       | 4                                       | 41                                            | 94                                         | <b>146</b>   |
| Solved                              | 201                                     | 209                                     | 162                                           | <b>432</b>                                 | 424          |
| <b>Logic S4, varying domains</b>    |                                         |                                         |                                               |                                            |              |
| Theorem                             | -                                       | 169                                     | -                                             | <b>274</b>                                 | 245          |
| Non-Theorem                         | -                                       | 4                                       | -                                             | 119                                        | <b>184</b>   |
| Solved                              | -                                       | 173                                     | -                                             | 393                                        | <b>429</b>   |



## More Flexibility in HOML

## Many Variations of Higher-Order Modal Logics

- ▶ (Meta-)Axioms for the accessibility relation (e.g. reflexivity, transitivity, symmetry)
- ▶ Axioms (e.g.  $\Box A \rightarrow A$ ,  $\Box A \rightarrow \Box \Box A$ ,  $A \rightarrow \Box \Diamond A$ )
- ▶ Multimodal logics, bridge rules
- ▶ Combination with other (non-)classical logics
- ▶ Constant domains vs. varying domains vs. cumulative domains
- ▶ Rigidity vs. **Flexibility**
- ▶ Simple types vs. dependent types (e.g.  $\mu$  vs.  $\mu(w)$ )
- ▶ ...

## Flexibility

### A Funny Example

□*Blue*(*sky*)



# Flexibility

## A Funny Example

□*Blue*(*sky*)



## Flexibility

### A Funny Example

$\Box \text{Blue}(\text{sky})$



Human: "Earth's sky is blue"



Martian: "Mars' sky is blue"



$$\Box Blue(sky)$$

► Rigid embedding:

$$[\mu] = \mu \quad [o] = \iota \rightarrow o \quad [\alpha \rightarrow \beta] = [\alpha] \rightarrow [\beta]$$

$$[sky_\mu] = sky_\mu$$
$$[Blue_{\mu \rightarrow o}] = Blue_{\mu \rightarrow (\iota \rightarrow o)}$$

► Flexible embedding:

$$[\mu] = \iota \rightarrow \mu \quad [o] = \iota \rightarrow o \quad [\alpha \rightarrow \beta] = \iota \rightarrow [\alpha] \rightarrow [\beta]$$

$$[sky_\mu] = sky_{\iota \rightarrow \mu}$$
$$[Blue_{\mu \rightarrow o}] = Blue_{\iota \rightarrow (\iota \rightarrow \mu) \rightarrow (\iota \rightarrow o)}$$



# Flexibility

## Flexible Embedding is Ambiguous

$$[\Box Blue(sky)]$$

$$\forall w. \forall w'. Blue\ w\ (sky\ w)\ w'$$

$$\forall w. \forall w'. Blue\ w'\ (sky\ w')\ w'$$

$$\forall w. \forall w'. Blue\ w'\ (sky\ w)\ w'$$

$$\forall w. \forall w'. Blue\ w\ (sky\ w')\ w'$$

## Some References (selection; see also the references therein)

- ▶ Quantified Modal Logic
  - ▶ Montague's and Bressan's work
  - ▶ Intensional and Higher-Order Modal Logic (D. Gallin), North-Holland, 1975.
  - ▶ Types, Tableaus, and Gödels God, (M. Fitting), Kluwer, 2002.
  - ▶ Interpolation for First Order S5 (M. Fitting), JSL, 67(2):621-634, 2002.
  - ▶ [Higher-Order Modal Logic](#) (R. Muskens), Handbook of Modal Logic, 2007.
- ▶ Relevance for Metaphysics (recent monographs; see references therein)
  - ▶ Modal Logic as Metaphysics (T. Williamson), Oxford University Press, 2013,
  - ▶ Mere Possibilities: Metaphysical Foundations of Modal Semantics (R. Stalnaker), Princeton University Press, 2011.
  - ▶ ...
- ▶ Embedding in HOL (joint work with colleagues)
  - ▶ [Higher-Order Modal Logics: Automation and Applications](#), Reasoning Web, 2015.
  - ▶ [Quantified Multimodal Logics in Simple Type Theory](#), Logica Universalis, 2013.
  - ▶ [Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers](#), ECAI, 2014.
  - ▶ [Systematic Verification of the Modal Logic Cube in Isabelle/HOL](#), EPTCS, 2015.
  - ▶ [HOL based First-order Modal Logic Provers](#), LPAR, 2013.
  - ▶ [Implementing and Evaluating Provers for First-order Modal Logics](#), ECAI, 2012.

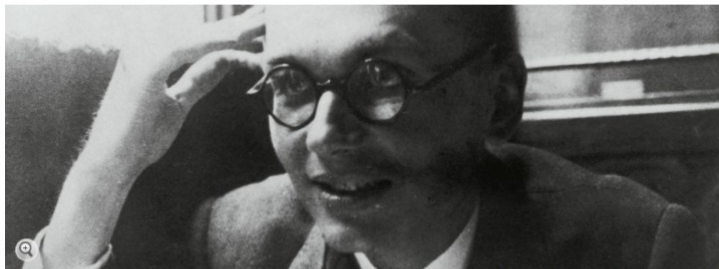
## Higher-Order Modal Logics (Lecture 4 - Part 1): Ontological Argument

Christoph Benz Müller and Bruno Woltzenlogel Paleo



## Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter



picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

**Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.**

Montag, 09.09.2013 - 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

3



## Germany

- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

## Austria

- Die Presse
- Wiener Zeitung
- ORF
- ...

## Italy

- Repubblica
- L'Espresso
- ...

## India

- DNA India
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- India Today
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- ABC News
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- Spiegel International
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- United Press Intl.
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## SCIENCE NEWS

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HOME / SCIENCE NEWS / RESEARCHERS SAY THEY USED MACBOOK TO PROVE GOEDEL'S GOD THEOREM

# Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | [1 comments](#)

See more serious and funny news links at  
`github.com/FormalTheology/GoedelGod/tree/master/Press`

# A Long History of Ontological Arguments

pros and cons



Anselm's notion of God (Proslogion, 1078):

**“God is that, than which nothing greater can be conceived.”**

Gödel's notion of God:

**“A God-like being possesses all ‘positive’ properties.”**

To show by logical, deductive reasoning:

**“God exists.”**

$$\exists xG(x)$$



# A Long History of Ontological Arguments

pros and cons



Anselm's notion of God (Proslogion, 1078):

**“God is that, than which nothing greater can be conceived.”**

Gödel's notion of God:

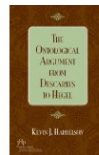
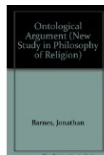
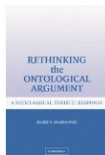
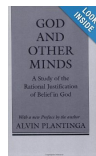
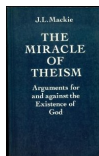
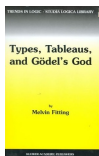
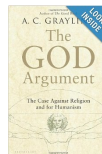
**“A God-like being possesses all ‘positive’ properties.”**

To show by logical, deductive reasoning:

**“Necessarily, God exists.”**

$$\Box \exists x G(x)$$

# The Ontological Proof Today



See also our collection of recent papers:

<https://github.com/FormalTheology/GoedelGod/tree/master/Literature>

C. Benz Müller and B. Woltzenlogel Paleo, 2015 — Higher-Order Modal Logics (Lecture 4)

## Various Different Interests in Ontological Arguments

- ▶ **Theistic:** persuasion of atheists of God's existence
- ▶ **Philosophical:** development of metaphysical theories
  - ▶ Meta-issues: metaphysical necessity vs. modal logical necessity
- ▶ **Our Goals:** can computers (theorem provers) be used to . . .
  - . . . formalize the definitions, axioms and theorems?
  - . . . verify/falsify the arguments step-by-step?
  - . . . automate (sub-)arguments?
  - . . . discover new philosophical knowledge?

## Vision of Leibniz (1646–1716): *Calculus!*



If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . .dicere: *calculemus*.  
(Leibniz, 1684)



Required:  
**characteristica universalis** and **calculus ratiocinator**

# Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Beweis

Feb 10, 1970

P(φ) φ is positive (i.e. φ ∈ P)

At 1:  $P(φ) \cdot P(ψ) \supset P(φ \cdot ψ)$  At 2:  $P(φ) \vee P(ψ) \supset P(φ \vee ψ)$

P1  $G(x) \equiv (φ) [P(φ) \supset φ(x)]$  (God)

P2  $φ \text{ Ess } x \equiv (ψ) [ψ(x) \supset N(y)[φ(y) \supset ψ(y)]]$  (Essence of x)

$P \supset Nq = N(p \supset q)$  Necessity

At 2:  $P(φ) \supset NP(φ)$   
 $\sim P(φ) \supset N\sim P(φ)$  } because it follows from the nature of the property

Th.  $G(x) \supset G \text{ Ess } x$

Df.  $E(x) \equiv (φ)[φ \text{ Ess } x \supset N\exists x φ(x)]$  necessary Existence

Ax 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(\exists x) G(x) \supset MN(\exists y) G(y)$

"  $\supset N(\exists y) G(y)$

M = possibility

any two essences of x are nec. equivalent

exclusive or \* and for any number of humanoids

$M(\exists x) G(x)$  means <sup>the system of</sup> all pos. props. is. com-feasible. This is true because of:

At 4:  $P(φ) \cdot φ \supset_N ψ \supset P(ψ)$  which impl.

~~then~~  $\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. props. were incomp. It would mean that the comp. prop. is (which is positive) would be  $x \neq x$

Positive means positive in the modal aesthetic sense (independently of the accidental structure of the world). Only then the attr. time. It also means "attribution" as opposed to "privation" (or containing privation). This is the point.

If  $φ$  is a pos. prop.  $(x) N \sim φ(x)$  then  $φ(x) \supset N \sim φ(x)$

hence  $x \neq x$  positive.  $x=x$  is negative. At 4

x i.e. the normal form in terms of elem. prop. contains no member without negation.

## Proof Overview

**T3:**  $\Box \exists x. G(x)$

## Proof Overview

**C1:**  $\Diamond \exists z. G(z)$

---

**T3:**  $\Box \exists x. G(x)$

## Proof Overview

$$\frac{\mathbf{C1:} \Diamond \exists z. G(z) \quad \mathbf{L2:} \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \Box \exists x. G(x)}$$



## Proof Overview

**L2:**  $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

**C1:**  $\Diamond \exists z. G(z)$       **L2:**  $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

---

**T3:**  $\Box \exists x. G(x)$

## Proof Overview

$$\frac{\frac{\mathbf{C1}: \Diamond \exists z.G(z) \quad \mathbf{L2}: \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\mathbf{T3}: \Box \exists x.G(x)} \quad \frac{\mathbf{L2}: \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\forall \bar{\xi} . [\Diamond \Box \bar{\xi} \rightarrow \Box \bar{\xi}] \quad \mathbf{S5}}$$

# Proof Overview

$$\begin{array}{c}
 \begin{array}{c}
 \diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x) \\
 \hline
 \mathbf{L2:} \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{S5} \\
 \forall \xi. [\Box \xi \rightarrow \Box \xi]
 \end{array}
 \\
 \hline
 \begin{array}{c}
 \mathbf{C1:} \diamond \exists z. G(z) \quad \mathbf{L2:} \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{T3:} \Box \exists x. G(x)
 \end{array}
 \end{array}$$

# Proof Overview

$$\begin{array}{c}
 \textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
 \hline
 \Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x) \qquad \textbf{S5} \quad \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi] \\
 \hline
 \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)
 \end{array}$$

$$\begin{array}{c}
 \textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x) \\
 \hline
 \textbf{T3: } \Box \exists x.G(x)
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\frac{\mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x) \quad \overline{\forall \xi. [\Box \xi \rightarrow \xi]} \quad \mathbf{S5}}{\diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \quad \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$$

$$\frac{\mathbf{C1:} \ \diamond \exists z. G(z) \quad \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\begin{array}{c}
 \frac{}{P(E)} \\
 \hline
 \frac{\mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \frac{\mathbf{S5} \quad \bar{\forall} \bar{\xi}. [\bar{\Diamond} \bar{\Box} \bar{\xi} \rightarrow \bar{\Box} \bar{\xi}]}{\bar{\forall} \bar{\xi}. [\bar{\Diamond} \bar{\Box} \bar{\xi} \rightarrow \bar{\Box} \bar{\xi}]} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y) \text{ (cheating!)}$$

$$\begin{array}{c}
 \frac{}{P(E)} \\
 \hline
 \frac{\mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \frac{\mathbf{S5} \quad \bar{\forall} \bar{\xi}. [\bar{\Diamond} \bar{\Box} \bar{\xi} \rightarrow \bar{\Box} \bar{\xi}]}{\bar{\forall} \bar{\xi}. [\bar{\Diamond} \bar{\Box} \bar{\xi} \rightarrow \bar{\Box} \bar{\xi}]} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y] \qquad P(E) \\
 \hline
 \mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x) \qquad \mathbf{S5} \quad \overline{\forall \xi. [\Box \xi \rightarrow \Box \xi]} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \\
 \mathbf{C1:} \ \Diamond \exists z. G(z) \qquad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{T3:} \ \Box \exists x. G(x)
 \end{array}$$



## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y] \qquad \frac{\mathbf{A5}}{P(E)} \\
 \hline
 \mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x) \qquad \frac{\mathbf{S5}}{\forall \xi. [\diamond \Box \xi \rightarrow \Box \xi]} \\
 \hline
 \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{C1:} \ \diamond \exists z. G(z) \qquad \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{T3:} \ \Box \exists x. G(x)
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y] \qquad \frac{\mathbf{A5}}{P(E)} \\
 \hline
 \mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x) \qquad \frac{\mathbf{S5}}{\forall \xi. [\diamond \Box \xi \rightarrow \Box \xi]} \\
 \hline
 \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{C1:} \ \diamond \exists z. G(z) \qquad \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{T3:} \ \Box \exists x. G(x)
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D2:} \ \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \frac{\frac{\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \overline{\forall \varphi. [P(\varphi) \longrightarrow \Box P(\varphi)]}}{\mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \frac{\overline{P(E)}}{\mathbf{A5}} \\
 \frac{\mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x)} \quad \frac{\overline{\forall \xi. [\diamond \Box \xi \rightarrow \Box \xi]}}{\mathbf{S5}} \\
 \frac{\diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x)}{\mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \\
 \frac{\mathbf{C1:} \ \diamond \exists z. G(z) \quad \mathbf{L2:} \ \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D2:} \ \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\mathbf{C1:} \ \Diamond \exists z. G(z)$$

$$\begin{array}{c}
 \frac{\frac{\overline{\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}} \quad \overline{\overline{\forall \varphi. [P(\varphi) \longrightarrow \Box P(\varphi)]}}}{\mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y]} \quad \overline{\overline{P(E)}} \quad \mathbf{A5}}{\frac{\mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)}} \quad \overline{\overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}} \quad \mathbf{S5}}{\mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)} \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

## Proof Overview

$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D2: } \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3^*: } E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3: } E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \hline P(G) \\
 \hline \mathbf{C1: } \Diamond \exists z. G(z) \\
 \\
 \begin{array}{ccc}
 \frac{\overline{\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}}}{\mathbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]} & \frac{\overline{\overline{\forall \varphi. [P(\varphi) \longrightarrow \Box P(\varphi)]}}}{\mathbf{A4}} & \frac{\overline{\overline{P(E)}}}{\mathbf{A5}} \\
 \hline
 \frac{\mathbf{L1: } \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} & & \frac{\overline{\overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}}}{\mathbf{S5}} \\
 \hline
 \mathbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \\
 \frac{\mathbf{C1: } \Diamond \exists z. G(z) \quad \mathbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3: } \Box \exists x. G(x)} \\
 \hline
 \end{array}
 \end{array}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D2:**  $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$

**D3\*:**  $E(x) \equiv \Box \exists y.G(y)$

**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$

$$\frac{A3}{P(G)}$$

**C1:**  $\Diamond \exists z. G(z)$

**A1b**  
 $\overline{\forall \varphi} . [\overline{\neg P(\varphi)} \rightarrow \overline{P(\neg \varphi)}]$

**A4**  
 $\overline{\forall \varphi} . [\overline{P(\varphi)} \longrightarrow \overline{\Box P(\varphi)}]$

     **A5**       
      $P(E)$      

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

$$\mathbf{L1:} \exists z.G(z) \rightarrow \Box \exists x.G(x)$$

$$\overline{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)}$$

**S5**  
 $\overline{\forall \xi} . [\overline{\Diamond \Box \xi} \rightarrow \overline{\Box \xi}]$

**L2:**  $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

**C1:**  $\Diamond \exists z. G(z)$

**L2:**  $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

**T3:**  $\Box \exists x. G(x)$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D2:} \ \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \frac{\mathbf{A3} \quad \overline{P(G)}}{\overline{P(G)}} \qquad \mathbf{T1:} \ \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)] \\
 \hline
 \mathbf{C1:} \ \Diamond \exists z. G(z) \\
 \\
 \frac{\frac{\mathbf{A1b} \quad \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}}{\overline{\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]}} \quad \mathbf{A4} \quad \overline{\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]} \quad \mathbf{A5} \quad \overline{P(E)}}{\mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y]} \\
 \hline
 \frac{\mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \mathbf{S5} \quad \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

## Proof Overview

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D2:} \ \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3^*}: \ E(x) \equiv \Box \exists y. G(y)$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \begin{array}{c}
 \mathbf{A3} \\
 \hline
 \overline{P(G)}
 \end{array}
 \quad
 \frac{
 \begin{array}{c}
 \mathbf{A2} \\
 \hline
 \overline{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \Box \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A1a} \\
 \hline
 \overline{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}
 \end{array}
 }{
 \mathbf{T1:} \ \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]
 } \\
 \hline
 \mathbf{C1:} \ \Diamond \exists z. G(z) \\
 \\
 \begin{array}{c}
 \mathbf{A1b} \\
 \hline
 \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A4} \\
 \hline
 \overline{\forall \varphi. [P(\varphi) \longrightarrow \Box P(\varphi)]}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A5} \\
 \hline
 \overline{P(E)}
 \end{array} \\
 \hline
 \mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y] \\
 \hline
 \begin{array}{c}
 \mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{S5} \\
 \hline
 \overline{\forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}
 \end{array} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \\
 \begin{array}{c}
 \mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline
 \mathbf{T3:} \ \Box \exists x. G(x)
 \end{array}
 \end{array}$$



## Proof Overview

$$\mathbf{D1: } G(x) \equiv \forall \varphi . [P(\varphi) \longrightarrow \varphi(x)]$$

$$\mathbf{D2: } \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi . (\psi(x) \rightarrow \Box \forall x . (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3: } NE(x) \equiv \forall \varphi . [\varphi \text{ ess } x \rightarrow \Box \exists y . \varphi(y)]$$

$$\begin{array}{c}
 \frac{\frac{\mathbf{A3}}{P(\overline{G})} \quad \frac{\frac{\mathbf{A2}}{\forall \varphi . \forall \psi . [(P(\varphi) \wedge \Box \forall x . [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \quad \frac{\mathbf{A1a}}{\forall \varphi . [P(\neg \varphi) \rightarrow \neg P(\varphi)]}}{\mathbf{T1: } \forall \varphi . [P(\varphi) \rightarrow \Diamond \exists x . \varphi(x)]} \\
 \hline
 \mathbf{C1: } \Diamond \exists z . G(z) \\
 \\
 \frac{\frac{\mathbf{A1b}}{\forall \varphi . [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \frac{\mathbf{A4}}{\forall \varphi . [P(\varphi) \longrightarrow \Box P(\varphi)]}}{\mathbf{T2: } \forall y . [G(y) \rightarrow G \text{ ess } y]} \quad \frac{\mathbf{A5}}{P(\overline{NE})} \\
 \hline
 \frac{\mathbf{L1: } \exists z . G(z) \rightarrow \Box \exists x . G(x)}{\Diamond \exists z . G(z) \rightarrow \Diamond \Box \exists x . G(x)} \quad \frac{\mathbf{S5}}{\forall \xi . [\Diamond \Box \xi \rightarrow \Box \xi]} \\
 \hline
 \mathbf{L2: } \Diamond \exists z . G(z) \rightarrow \Box \exists x . G(x) \\
 \\
 \frac{\mathbf{C1: } \Diamond \exists z . G(z) \quad \mathbf{L2: } \Diamond \exists z . G(z) \rightarrow \Box \exists x . G(x)}{\mathbf{T3: } \Box \exists x . G(x)}
 \end{array}$$

## Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall \varphi [P(\neg \varphi) \leftrightarrow \neg P(\varphi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\forall \varphi \forall \psi [(P(\varphi) \wedge \Box \forall x [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm. T1** Positive properties are possibly exemplified:

$$\forall \varphi [P(\varphi) \rightarrow \Diamond \exists x \varphi(x)]$$

**Def. D1** A God-like being possesses all positive properties:

$$G(x) \leftrightarrow \forall \varphi [P(\varphi) \rightarrow \varphi(x)]$$

**Axiom A3** The property of being God-like is positive:

$$P(G)$$

**Cor. C** Possibly, God exists:

$$\Diamond \exists x G(x)$$

**Axiom A4** Positive properties are necessarily positive:

$$\forall \varphi [P(\varphi) \rightarrow \Box P(\varphi)]$$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:  $\varphi \text{ ess } x \leftrightarrow \varphi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\varphi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:

$$\forall x [G(x) \rightarrow G \text{ ess } x]$$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall \varphi [\varphi \text{ ess } x \rightarrow \Box \exists y \varphi(y)]$$

**Axiom A5** Necessary existence is a positive property:

$$P(NE)$$

**Thm. T3** Necessarily, God exists:

$$\Box \exists x G(x)$$

# Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$

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$$G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

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$$P(G)$$

**Cor. C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**Axiom A4** Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box P(\varphi)]$$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi \text{ ess } x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

**Thm. T2** Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \text{ ess } x]$$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\varphi[\varphi \text{ ess } x \rightarrow \Box\exists y\varphi(y)]$$

**Axiom A5** Necessary existence is a positive property:

$$P(NE)$$

**Thm. T3** Necessarily, God exists:

$$\Box\exists xG(x)$$

Difference to Gödel (who omits this conjunct)

# Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \boxed{\forall x}[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm. T1** Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

**Def. D1** A God-like being possesses all positive properties:

$$G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

**Axiom A3** The property of being God-like is positive:

$$P(G)$$

**Cor. C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**Axiom A4** Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \boxed{\forall x}P(\varphi)]$$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:  $\varphi \text{ ess } x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \boxed{\forall y}(\varphi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \text{ ess } x]$$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\varphi[\varphi \text{ ess } x \rightarrow \boxed{\exists y}\varphi(y)]$$

**Axiom A5** Necessary existence is a positive property:

$$P(NE)$$

**Thm. T3** Necessarily, God exists:

$$\boxed{\exists x}G(x)$$

Modal operators are used

## Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\boxed{\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}$$

**Thm. T1** Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

**Def. D1** A God-like being possesses all positive properties:

$$G(x) \leftrightarrow \boxed{\forall\varphi[P(\varphi) \rightarrow \varphi(x)]}$$

**Axiom A3** The property of being God-like is positive:

$$P(G)$$

**Cor. C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**Axiom A4** Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box P(\varphi)]$$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:  $\varphi \text{ ess } x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \text{ ess } x]$$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\varphi[\varphi \text{ ess } x \rightarrow \Box\exists y\varphi(y)]$$

**Axiom A5** Necessary existence is a positive property:

$$P(NE)$$

**Thm. T3** Necessarily, God exists:

$$\Box\exists xG(x)$$

second-order quantifiers

# The Ontological Argument in TPTP THF0

```
1  %-----
2  %---Axioms for Quantified Modal Logic KB.
3  include('Quantified_KB.ax').
4  %-----
5  %---constant symbol for positive (p), God-like (g), essence (ess), necessary existence (ne)
6  thf(p_tp,type,(p:(mu>$i>$o)>$i>$o)).
7  thf(g_tp,type,(g:(mu>$i>$o)).
8  thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).
9  thf(ne_tp,type,(ne:(mu>$i>$o)).
10 %---D1:A God-like being possesses all positive properties.
11 thf(defD1,definition,(g = (^[X:mu]:(mforall_indset^[Phi:mu>$i>$o]:(mimplies@(p@Phi)@(Phi@X)))))).
12 %---C: Possibly, God exists. (Proved in C.p)
13 thf(corC,axiom,(v@(mdia@(mexists_ind^[X:mu]:(g@X))))).
14 %---T2: Being God-like is an essence of any God-like being. (Proved in T2.p)
15 thf(thmT2,axiom,(v@(mforall_ind^[X:mu]:(mimplies@(g@X)@(ess@g@X))))).
16 %---D3: Necessary existence of an individual is the necessary exemplification of all its essences
17 thf(defD3,definition,(ne = (^[X:mu]:(mforall_indset^[Phi:mu>$i>$o]:
18   (mimplies@(ess@Phi@X)@(mbox@(mexists_ind^[Y:mu]:(Phi@Y))))))).
19 %---A5:Necessary existence is positive.
20 thf(axA5,axiom,(v@(p@ne))).
21 %---T3: Necessarily God exists.
22 thf(thmT3,conjecture,(v@(mbox@(mexists_ind^[X:mu]:(g@X))))).
```

## The Ontological Argument in Isabelle/HOL

```
GoedelGod.thy (-/Varia/Talks/2015-ReasoningWeb/DemoMaterial/DEMO-Isabelle/)
text {* QML formulas are translated as HOL terms of type  $\text{@(typ } "i \Rightarrow \text{bool} \text{")}$ .
This type is abbreviated as  $\text{@(text } "\sigma \text{")}$ . *}

type_synonym  $\sigma = "(i \Rightarrow \text{bool})"$ 

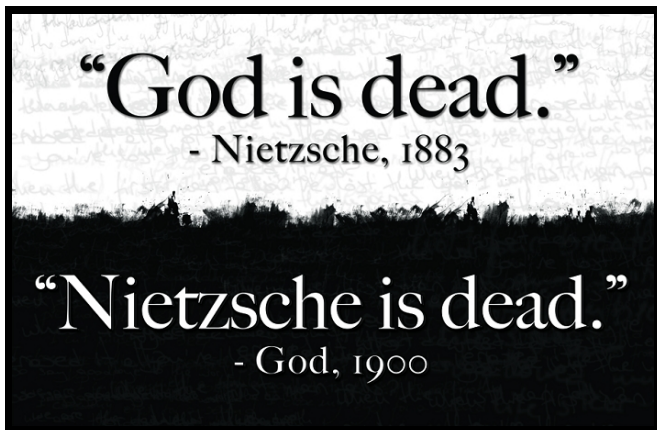
text {* The classical connectives  $\neg$ ,  $\wedge$ ,  $\rightarrow$ , and  $\forall$  (over individuals and over sets of individuals) are
lifted to type  $\sigma$ . The lifted connectives are  $\text{@(text } "m\text{--}" \text{)}$ ,  $\text{@(text } "m\wedge" \text{)}$ ,  $\text{@(text } "m\rightarrow" \text{)}$ ,
 $\text{@(text } "m\forall" \text{)}$ , and  $\text{@(text } "m\exists" \text{)}$  (the latter two are modeled as constant symbols).
Other connectives can be introduced analogously. We exemplarily do this for  $\text{@(text } "m\vee" \text{)}$ ,
 $\text{@(text } "m\equiv" \text{)}$ , and  $\text{@(text } "mL=" \text{)}$  (Leibniz equality on individuals). Moreover, the modal
operators  $\text{@(text } "\Box" \text{)}$  and  $\text{@(text } "\Diamond" \text{)}$  are introduced. Definitions could be used instead of
abbreviations. *}

abbreviation mnot :: " $\sigma \Rightarrow \sigma$ " ( $"m\text{--}"$ ) where  $"m\text{--} \varphi \equiv (\lambda w. \neg \varphi w)"$ 
abbreviation mand :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " ( $"m\wedge"$ ) 51 where  $"\varphi m\wedge \psi \equiv (\lambda w. \varphi w \wedge \psi w)"$ 
abbreviation mor :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " ( $"m\vee"$ ) 50 where  $"\varphi m\vee \psi \equiv (\lambda w. \varphi w \vee \psi w)"$ 
abbreviation mimplies :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " ( $"m\rightarrow"$ ) 49 where  $"\varphi m\rightarrow \psi \equiv (\lambda w. \varphi w \longrightarrow \psi w)"$ 
abbreviation mequiv :: " $\sigma \Rightarrow \sigma \Rightarrow \sigma$ " ( $"m\equiv"$ ) 48 where  $"\varphi m\equiv \psi \equiv (\lambda w. \varphi w \longleftrightarrow \psi w)"$ 
abbreviation mforall :: " $(\text{'a} \Rightarrow \sigma) \Rightarrow \sigma$ " ( $"m\forall"$ ) where  $"\forall \Phi \equiv (\lambda w. \forall x. \Phi x w)"$ 
abbreviation mexists :: " $(\text{'a} \Rightarrow \sigma) \Rightarrow \sigma$ " ( $"m\exists"$ ) where  $"\exists \Phi \equiv (\lambda w. \exists x. \Phi x w)"$ 
abbreviation mLeibeq :: " $\mu \Rightarrow \mu \Rightarrow \sigma$ " ( $"mL="$ ) 52 where  $"x mL= y \equiv \forall (\lambda \varphi. (\varphi x m\rightarrow \varphi y))"$ 
abbreviation mbox :: " $\sigma \Rightarrow \sigma$ " ( $"\Box"$ ) where  $"\Box \varphi \equiv (\lambda w. \forall v. w r v \longrightarrow \varphi v)"$ 
abbreviation mdia :: " $\sigma \Rightarrow \sigma$ " ( $"\Diamond"$ ) where  $"\Diamond \varphi \equiv (\lambda w. \exists v. w r v \wedge \varphi v)"$ 

text {* For grounding lifted formulas, the meta-predicate  $\text{@(text } "valid" \text{)}$  is introduced. *}

(*<*) no_syntax "_list" :: " $\text{args} \Rightarrow \text{'a list}$ " ( $"[\_]"$ ) (*>*)
abbreviation valid :: " $\sigma \Rightarrow \text{bool}$ " ( $"[\_]"$ ) where  $"[p] \equiv \forall w. p w"$ 
```

See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:  
<http://afp.sourceforge.net/entries/GoedelGod.shtml>



## **Experimental Results and Philosophical Discoveries**

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]



# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                                                                            | dependencies                                                                                               | logic              | status                   | LEO-II<br>const/vary         | Satallax<br>const/vary       | Nitpick<br>const/vary            |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|--------------------|--------------------------|------------------------------|------------------------------|----------------------------------|
| A1  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \neg(\phi X)) \dot{=} \neg(p\phi)]$                                                                                                                                               |                                                                                                            |                    |                          |                              |                              |                                  |
| A2  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} \check{\forall}\psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\exists}\check{\forall}X_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$                                                 |                                                                                                            |                    |                          |                              |                              |                                  |
| T1  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\exists}\check{\forall}X_{\mu} \cdot \phi X]$                                                                                                                                    | A1( $\supset$ ), A2<br>A1, A2                                                                              | K<br>K             | THM<br>THM               | 0.1/0.1<br>0.1/0.1           | 0.0/0.0<br>0.0/5.2           | —/—<br>—/—                       |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                             |                                                                                                            |                    |                          |                              |                              |                                  |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                          |                                                                                                            |                    |                          |                              |                              |                                  |
| C   | $[\dot{\exists}\check{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                              | T1, D1, A3<br>A1, A2, D1, A3                                                                               | K<br>K             | THM<br>THM               | 0.0/0.0<br>0.0/0.0           | 0.0/0.0<br>5.2/31.3          | —/—<br>—/—                       |
| A4  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} p\phi]$                                                                                                                                                                               |                                                                                                            |                    |                          |                              |                              |                                  |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \phi X \wedge \check{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\exists}\check{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                                                                                            |                    |                          |                              |                              |                                  |
| T2  | $[\check{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                                              | A1, D1, A4, D2<br>A1, A2, D1, A3, A4, D2                                                                   | K<br>K             | THM<br>THM               | 19.1/18.3<br>12.9/14.0       | 0.0/0.0<br>0.0/0.0           | —/—<br>—/—                       |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \check{\forall}\phi_{\mu \rightarrow \sigma} \cdot (\text{ess } \phi X \dot{\supset} \dot{\exists}\check{\forall}Y_{\mu} \cdot \phi Y)$                                                                                                     |                                                                                                            |                    |                          |                              |                              |                                  |
| A5  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                  |                                                                                                            |                    |                          |                              |                              |                                  |
| T3  | $[\dot{\exists}\check{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                              | D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5<br>D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5 | K<br>K<br>KB<br>KB | CSA<br>CSA<br>THM<br>THM | —/—<br>—/—<br>0.0/0.1<br>—/— | —/—<br>—/—<br>0.1/5.3<br>—/— | 3.8/6.2<br>8.2/7.5<br>—/—<br>—/— |
| MC  | $[s_{\sigma} \dot{\supset} \dot{\exists}s_{\sigma}]$                                                                                                                                                                                                                                                    | D2, T2, T3<br>A1, A2, D1, A3, A4, D2, D3, A5                                                               | KB<br>KB           | THM<br>THM               | 17.9/—<br>—/—                | 3.3/3.2<br>—/—               | —/—<br>—/—                       |
| FG  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} \check{\forall}X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (\neg(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\supset} \neg(\phi X)))]$                                                                                      | A1, D1<br>A1, A2, D1, A3, A4, D2, D3, A5                                                                   | KB<br>KB           | THM<br>THM               | 16.5/—<br>12.8/15.1          | 0.0/0.0<br>0.0/5.4           | —/—<br>—/—                       |
| MT  | $[\check{\forall}X_{\mu} \cdot \check{\forall}Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} X \dot{=} Y))]$                                                                                                                                     | D1, FG<br>A1, A2, D1, A3, A4, D2, D3, A5                                                                   | KB<br>KB           | THM<br>THM               | —/—<br>—/—                   | 0.0/3.3<br>—/—               | —/—<br>—/—                       |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                            | A1, A2, D1, A3, A4, D2, D3, A5                                                                             | KB                 | SAT                      | —/—                          | —/—                          | 7.3/7.4                          |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \check{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\exists}\check{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$               |                                                                                                            |                    |                          |                              |                              |                                  |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                            | A1( $\supset$ ), A2, D2', D3, A5<br>A1, A2, D1, A3, A4, D2', D3, A5                                        | KB<br>KB           | UNS<br>UNS               | 7.5/7.8<br>—/—               | —/—<br>—/—                   | —/—<br>—/—                       |

# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                                                                     | dependencies                                                                                               | logic              | status                   | LEO-II<br>const/vary         | Satallax<br>const/vary       | Nitpick<br>const/vary            |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|--------------------|--------------------------|------------------------------|------------------------------|----------------------------------|
| A1  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$                                                                                                                       |                                                                                                            |                    |                          |                              |                              |                                  |
| A2  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} \check{\forall}\psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} X_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$                                                  |                                                                                                            |                    |                          |                              |                              |                                  |
| T1  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\exists} X_{\mu} \cdot \phi X]$                                                                                                                                           | A1( $\supset$ ), A2<br>A1, A2                                                                              | K<br>K             | THM<br>THM               | 0.1/0.1<br>0.1/0.1           | 0.0/0.0<br>0.0/5.2           | —/—<br>—/—                       |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                      |                                                                                                            |                    |                          |                              |                              |                                  |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                   |                                                                                                            |                    |                          |                              |                              |                                  |
| C   | $[\dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                     | T1, D1, A3<br>A1, A2, D1, A3                                                                               | K<br>K             | THM<br>THM               | 0.0/0.0<br>0.0/0.0           | 0.0/0.0<br>5.2/31.3          | —/—<br>—/—                       |
| A4  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} p\phi]$                                                                                                                                                                        |                                                                                                            |                    |                          |                              |                              |                                  |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \phi X \dot{\wedge} \check{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\forall} Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                                                                                            |                    |                          |                              |                              |                                  |
| T2  | $[\check{\forall} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                                      | A1, D1, A4, D2<br>A1, A2, D1, A3, A4, D2                                                                   | K<br>K             | THM<br>THM               | 19.1/18.3<br>12.9/14.0       | 0.0/0.0<br>0.0/0.0           | —/—<br>—/—                       |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \check{\forall}\phi_{\mu \rightarrow \sigma} \cdot (\text{ess } \phi X \dot{\supset} \dot{\exists} Y_{\mu} \cdot \phi Y)$                                                                                                            |                                                                                                            |                    |                          |                              |                              |                                  |
| A5  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                           |                                                                                                            |                    |                          |                              |                              |                                  |
| T3  | $[\dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                     | D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5<br>D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5 | K<br>K<br>KB<br>KB | CSA<br>CSA<br>THM<br>THM | —/—<br>—/—<br>0.0/0.1<br>—/— | —/—<br>—/—<br>0.1/5.3<br>—/— | 3.8/6.2<br>8.2/7.5<br>—/—<br>—/— |
| MC  | $[s_{\sigma} \dot{\supset} \dot{\exists} s_{\sigma}]$                                                                                                                                                                                                                                            | D2, T2, T3<br>A1, A2, D1, A3, A4, D2, D3, A5                                                               | KB<br>KB           | THM<br>THM               | 17.9/—<br>—/—                | 3.3/3.2<br>—/—               | —/—<br>—/—                       |
| FG  | $[\check{\forall}\phi_{\mu \rightarrow \sigma} \check{\forall} X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\supset} \dot{\neg}(\phi X)))]$                                                                  | A1, D1<br>A1, A2, D1, A3, A4, D2, D3, A5                                                                   | KB<br>KB           | THM<br>THM               | 16.5/—<br>12.8/15.1          | 0.0/0.0<br>0.0/5.4           | —/—<br>—/—                       |
| MT  | $[\check{\forall} X_{\mu} \cdot \check{\forall} Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} X \dot{\equiv} Y))]$                                                                                                                       | D1, FG<br>A1, A2, D1, A3, A4, D2, D3, A5                                                                   | KB<br>KB           | THM<br>THM               | —/—<br>—/—                   | 0.0/3.3<br>—/—               | —/—<br>—/—                       |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                     | A1, A2, D1, A3, A4, D2, D3, A5                                                                             | KB                 | SAT                      | —/—                          | —/—                          | 7.3/7.4                          |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \check{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\forall} Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$                     |                                                                                                            |                    |                          |                              |                              |                                  |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                     | A1( $\supset$ ), A2, D2', D3, A5<br>A1, A2, D1, A3, A4, D2', D3, A5                                        | KB<br>KB           | UNS<br>UNS               | 7.5/7.8<br>—/—               | —/—<br>—/—                   | —/—<br>—/—                       |

# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                                                                               | dependencies           | logic | status | LEO-II<br>const/vary | Satallax<br>const/vary | Nitpick<br>const/vary |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------|-------|--------|----------------------|------------------------|-----------------------|
| A1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$                                                                                                                                   |                        |       |        |                      |                        |                       |
| A2  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \dot{\forall}\psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \psi \wedge \dot{\neg}\lambda X_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\phi]$                                                                  |                        |       |        |                      |                        |                       |
| T1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond}\exists X_{\mu} \cdot \phi X]$                                                                                                                                               | A1( $\supset$ ), A2    | K     | THM    | 0.1/0.1              | 0.0/0.0                | —/—                   |
|     |                                                                                                                                                                                                                                                                                                            | A1, A2                 | K     | THM    | 0.1/0.1              | 0.0/5.2                | —/—                   |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                                  |                        |       |        |                      |                        |                       |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                             |                        |       |        |                      |                        |                       |
| C   | $[\dot{\diamond}\exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                       | T1, D1, A3             | K     | THM    | 0.0/0.0              | 0.0/0.0                | —/—                   |
|     |                                                                                                                                                                                                                                                                                                            | A1, A2, D1, A3         | K     | THM    | 0.0/0.0              | 5.2/31.3               | —/—                   |
| A4  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond}p\phi]$                                                                                                                                                                      |                        |       |        |                      |                        |                       |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \phi X \wedge \dot{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\diamond}\dot{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$       |                        |       |        |                      |                        |                       |
| T2  | $[\dot{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} gX)]$                                                                                                                                                    | A1, D1, A4, D2         | K     | THM    | 19.1/18.3            | 0.0/0.0                | —/—                   |
|     |                                                                                                                                                                                                                                                                                                            | A1, A2, D1, A3, A4, D2 | K     | THM    | 12.9/14.0            | 0.0/0.0                | —/—                   |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot (e_{\mu \rightarrow \sigma} \phi X \dot{\supset} \dot{\diamond}\phi X)$                                                                                                                       |                        |       |        |                      |                        |                       |
| A5  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                     |                        |       |        |                      |                        |                       |
| T3  | $[\dot{\diamond}\exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                       |                        |       |        |                      |                        |                       |
| MC  | $[s_{\sigma} \dot{\supset} \dot{\diamond}s_{\sigma}]$                                                                                                                                                                                                                                                      |                        |       |        |                      |                        |                       |
| FG  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot \dot{\forall}X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} \dot{\diamond}\phi X)]$                                                                                                                                                          |                        |       |        |                      |                        |                       |
| MT  | $[\dot{\forall}X_{\mu} \cdot \dot{\forall}Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} \dot{\diamond}\phi X))]$                                                                                                                                   |                        |       |        |                      |                        |                       |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                               |                        |       |        |                      |                        |                       |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \phi X \wedge \dot{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\diamond}\dot{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                        |       |        |                      |                        |                       |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                               |                        |       |        |                      |                        |                       |

## Automating Scott's proof script

**T1: "Positive properties are possibly exemplified"**  
proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1 and A2
  - ▶ A1( $\supset$ ) and A2
- ▶ notion of quantification
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                                                                           | dependencies                             | logic | status | LEO-II<br>const/vary   | Satallax<br>const/vary | Nitpick<br>const/vary |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------|-------|--------|------------------------|------------------------|-----------------------|
| A1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$                                                                                                                         |                                          |       |        |                        |                        |                       |
| A2  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot \dot{\forall}\psi_{\mu \rightarrow \sigma} \cdot (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\neg}\dot{\forall}X_{\mu} \cdot (\phi X \dot{\supset} p\psi)) \dot{\supset} p\psi]$                                              |                                          |       |        |                        |                        |                       |
| T1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond}\dot{\exists}X_{\mu} \cdot \phi X]$                                                                                                                                | A1( $\supset$ ), A2                      | K     | THM    | 0.1/0.1                | 0.0/0.0                | —/—                   |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                        | A1, A2                                   | K     | THM    | 0.1/0.1                | 0.0/5.2                | —/—                   |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                         |                                          |       |        |                        |                        |                       |
| C   | $[\dot{\diamond}\dot{\exists}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                              | T1, D1, A3<br>A1, A2, D1, A3             | K     | THM    | 0.0/0.0                | 0.0/0.0                | —/—                   |
| A4  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\square}p\phi]$                                                                                                                                                             |                                          | K     | THM    | 0.0/0.0                | 5.2/31.3               | —/—                   |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \phi X \wedge \dot{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\neg}\dot{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                          |       |        |                        |                        |                       |
| T2  | $[\dot{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                                               | A1, D1, A4, D2<br>A1, A2, D1, A3, A4, D2 | K     | THM    | 19.1/18.3<br>12.9/14.0 | 0.0/0.0                | —/—                   |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} \phi X)$                                                                                                                                      |                                          |       |        |                        |                        |                       |
| A5  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                 |                                          |       |        |                        |                        |                       |
| T3  | $[\dot{\neg}\dot{\diamond}\dot{\exists}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                    |                                          |       |        |                        |                        |                       |
| MC  | $[s_{\sigma} \dot{\supset} \dot{\neg}s_{\sigma}]$                                                                                                                                                                                                                                                      |                                          |       |        |                        |                        |                       |
| FG  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot \dot{\forall}X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} \phi X)]$                                                                                                                                                                    |                                          |       |        |                        |                        |                       |
| MT  | $[\dot{\forall}X_{\mu} \cdot \dot{\forall}Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} \phi X))]$                                                                                                                                             |                                          |       |        |                        |                        |                       |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                           |                                          |       |        |                        |                        |                       |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \phi X$                                                                                                                                                   |                                          |       |        |                        |                        |                       |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                           |                                          |       |        |                        |                        |                       |

## Automating Scott's proof script

**C: "Possibly, God exists"**  
proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ T1, D1, A3
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

|    | HOL encoding                                                                                                                                                                                                                                                        | dependencies           | logic | status | LEO-II<br>const/vary | Satallax<br>const/vary | Nitpick<br>const/vary |
|----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------|-------|--------|----------------------|------------------------|-----------------------|
| A1 | $[\forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \neg(\phi X)) \equiv \neg(p\phi)]$                                                                                                                   |                        |       |        |                      |                        |                       |
| A2 | $[\forall \phi_{\mu \rightarrow \sigma} \forall \psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \neg \forall X_{\mu} (\phi X \supset \psi X)) \supset p\psi]$                                                            |                        |       |        |                      |                        |                       |
| T1 | $[\forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \diamond \exists X_{\mu} \cdot \phi X]$                                                                                                                        | A1( $\supset$ ), A2    | K     | THM    | 0.1/0.1              | 0.0/0.0                | —/—                   |
|    |                                                                                                                                                                                                                                                                     | A1, A2                 | K     | THM    | 0.1/0.1              | 0.0/5.2                | —/—                   |
| D1 | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \phi X$                                                                                                      |                        |       |        |                      |                        |                       |
| A3 | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                      |                        |       |        |                      |                        |                       |
| C  | $[\diamond \exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                     | T1, D1, A3             | K     | THM    | 0.0/0.0              | 0.0/0.0                | —/—                   |
|    |                                                                                                                                                                                                                                                                     | A1, A2, D1, A3         | K     | THM    | 0.0/0.0              | 5.2/31.3               | —/—                   |
| A4 | $[\forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \diamond p\phi]$                                                                                                                                               |                        |       |        |                      |                        |                       |
| D2 | $ess_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \phi X \wedge \forall \psi_{\mu \rightarrow \sigma} (\psi X \supset \diamond \forall I_{\mu} \cdot (\phi I \supset \psi I))$ |                        |       |        |                      |                        |                       |
| T2 | $[\forall X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \supset (ess_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                              | A1, D1, A4, D2         | K     | THM    | 19.1/18.3            | 0.0/0.0                | —/—                   |
|    |                                                                                                                                                                                                                                                                     | A1, A2, D1, A3, A4, D2 | K     | THM    | 12.9/14.0            | 0.0/0.0                | —/—                   |

## Automating Scott's proof script

**T2: "Being God-like is an ess. of any God-like being"**  
proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1, D1, A4, D2
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                                                                     | dependencies                             | logic  | status     | LEO-II<br>const/vary   | Satallax<br>const/vary | Nitpick<br>const/vary |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------|--------|------------|------------------------|------------------------|-----------------------|
| A1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$                                                                                                                         |                                          |        |            |                        |                        |                       |
| A2  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \dot{\forall}\psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\neg}\psi_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$                                                             |                                          |        |            |                        |                        |                       |
| T1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond}\exists X_{\mu} \cdot \phi X]$                                                                                                                                     | A1( $\supset$ ), A2<br>A1, A2            | K<br>K | THM<br>THM | 0.1/0.1<br>0.1/0.1     | 0.0/0.0<br>0.0/5.2     | —/—<br>—/—            |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                        |                                          |        |            |                        |                        |                       |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                   |                                          |        |            |                        |                        |                       |
| C   | $[\dot{\diamond}\exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                             | T1, D1, A3<br>A1, A2, D1, A3             | K<br>K | THM<br>THM | 0.0/0.0<br>0.0/0.0     | 0.0/0.0<br>5.2/31.3    | —/—<br>—/—            |
| A4  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond}p\phi]$                                                                                                                                                            |                                          |        |            |                        |                        |                       |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \phi X \wedge \dot{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\neg}\dot{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                          |        |            |                        |                        |                       |
| T2  | $[\dot{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                                         | A1, D1, A4, D2<br>A1, A2, D1, A2, A4, D2 | K<br>K | THM<br>THM | 19.1/18.3<br>12.0/11.0 | 0.0/0.0<br>0.0/0.0     | —/—<br>—/—            |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} \cdot (e_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi X \dot{\supset} \phi X)$                                                                                                      |                                          |        |            |                        |                        |                       |
| A5  | $[e_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                           |                                          |        |            |                        |                        |                       |
| T3  | $[\dot{\diamond}\exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                             |                                          |        |            |                        |                        |                       |
| MC  | $[s_{\sigma} \dot{\supset} \dot{\diamond}s_{\sigma}]$                                                                                                                                                                                                                                            |                                          |        |            |                        |                        |                       |
| FG  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \dot{\forall}X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} \phi X)]$                                                                                                                                                                    |                                          |        |            |                        |                        |                       |
| MT  | $[\dot{\forall}X_{\mu} \cdot \dot{\forall}Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} \phi X \dot{\supset} \phi Y))]$                                                                                                                  |                                          |        |            |                        |                        |                       |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                     |                                          |        |            |                        |                        |                       |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \phi X \wedge \dot{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\neg}\dot{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                          |        |            |                        |                        |                       |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                     |                                          |        |            |                        |                        |                       |

## Automating Scott's proof script

T3: "Necessarily, God exists"  
proved by LEO-II and Satallax

- ▶ in logic: **KB**
- ▶ from assumptions:
  - ▶ D1, C, T2, D3, A5
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

For logic **K** we got a **countermodel** by Nitpick

# Main Findings

|    | HOL encoding                                                                                                                                                                                                                                                                                         | dependencies              | logic | status | LEO-II<br>const/vary | Satallax<br>const/vary | Nitpick<br>const/vary |
|----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------|-------|--------|----------------------|------------------------|-----------------------|
| A1 | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \neg(\phi X)) \dot{=} \neg(p\phi)]$                                                                                                                                            |                           |       |        |                      |                        |                       |
| A2 | $[\check{\forall}\phi_{\mu \rightarrow \sigma} \check{\forall}\psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\neg}\check{\forall}X_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$                                                 |                           |       |        |                      |                        |                       |
| T1 | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\neg}\dot{\exists}X_{\mu} \cdot \phi X]$                                                                                                                                      | A1( $\dot{\supset}$ ), A2 | K     | THM    | 0.1/0.1              | 0.0/0.0                | —/—                   |
|    |                                                                                                                                                                                                                                                                                                      | A1, A2                    | K     | THM    | 0.1/0.1              | 0.0/5.2                | —/—                   |
| D1 | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                          |                           |       |        |                      |                        |                       |
| A3 | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                       |                           |       |        |                      |                        |                       |
| C  | $[\dot{\neg}\dot{\exists}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                | T1, D1, A3                | K     | THM    | 0.0/0.0              | 0.0/0.0                | —/—                   |
|    |                                                                                                                                                                                                                                                                                                      | A1, A2, D1, A3            | K     | THM    | 0.0/0.0              | 5.2/31.3               | —/—                   |
| A4 | $[\check{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\neg}p\phi]$                                                                                                                                                                  |                           |       |        |                      |                        |                       |
| D2 | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda\phi_{\mu \rightarrow \sigma} \lambda X_{\mu} \cdot \phi X \wedge \check{\forall}\psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\neg}\check{\forall}Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                           |       |        |                      |                        |                       |
| T2 | $[\check{\forall}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                                           | A1, D1, A4, D2            | K     | THM    | 19.1/18.3            | 0.0/0.0                | —/—                   |
|    |                                                                                                                                                                                                                                                                                                      | A1, A2, D1, A3, A4, D2    | K     | THM    | 12.9/14.0            | 0.0/0.0                | —/—                   |
| D3 | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \check{\forall}\phi_{\mu \rightarrow \sigma} \cdot (e_{\mu \rightarrow \sigma} \phi X \dot{\supset} \phi X)$                                                                                                                             |                           |       |        |                      |                        |                       |
| A5 | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                               |                           |       |        |                      |                        |                       |
| T3 | $[\dot{\neg}\dot{\exists}X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                |                           |       |        |                      |                        |                       |

## Automating Scott's proof script

### Summary

- ▶ proof verified and automated
- ▶ KB is sufficient (criticized logic **S5 not needed!**)
- ▶ possibilist and actualist quantifiers (individuals)
- ▶ exact dependencies determined experimentally
- ▶ ATPs have found **alternative proofs**  
e.g. self-identity  $\lambda x(x = x)$  is not needed

# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                                                  | dependencies                                                                          | logic         | status            | LEO-II<br>const/vary  | Satallax<br>const/vary | Nitpick<br>const/vary |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------|-------------------|-----------------------|------------------------|-----------------------|
| A1  | $[\forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \neg(\phi X)) \equiv \neg(p\phi)]$                                                                                                                             |                                                                                       |               |                   |                       |                        |                       |
| A2  | $[\forall \phi_{\mu \rightarrow \sigma} \forall \psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \neg \forall X_{\mu} \cdot (\phi X \supset \psi X)) \supset p\psi]$                                                                |                                                                                       |               |                   |                       |                        |                       |
| T1  | $[\forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \diamond \exists Y_{\mu} \cdot \neg \phi Y]$                                                                                                                             | A1( $\supset$ ), A2                                                                   | K             | THM               | 0.1/0.1               | 0.0/0.0                | —/—                   |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi$                                                                                                                               |                                                                                       |               |                   |                       |                        |                       |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                |                                                                                       |               |                   |                       |                        |                       |
| C   | $[\diamond \exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                               |                                                                                       |               |                   |                       |                        |                       |
| A4  | $[\forall \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \diamond p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi]$                                                                                                          |                                                                                       |               |                   |                       |                        |                       |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \forall \psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \neg \psi X \supset \psi X)$ |                                                                                       |               |                   |                       |                        |                       |
| T2  | $[\forall X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \supset (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} X \supset \diamond \exists Y_{\mu} \cdot \neg X Y)]$                                                                                   |                                                                                       |               |                   |                       |                        |                       |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \forall \phi_{\mu \rightarrow \sigma} (e_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \supset \phi X)$                                                                                                      |                                                                                       |               |                   |                       |                        |                       |
| A5  | $[e_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                        |                                                                                       |               |                   |                       |                        |                       |
| T3  | $[\diamond \exists X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                               | A1, A2, D1, A3, A4, D2, D3, A5<br>D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5 | K<br>KB<br>KB | CSA<br>THM<br>THM | —/—<br>0.0/0.1<br>—/— | —/—<br>0.1/5.3<br>—/—  | 8.2/1.5<br>—/—<br>—/— |
| MC  | $[s_{\sigma} \supset \diamond s_{\sigma}]$                                                                                                                                                                                                                                    | D2, T2, T3<br>A1, A2, D1, A3, A4, D2, D3, A5                                          | KB<br>KB      | THM<br>THM        | 17.9/—<br>—/—         | 3.3/3.2<br>—/—         | —/—<br>—/—            |
| FG  | $[\forall \phi_{\mu \rightarrow \sigma} \forall X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \supset (\neg(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \supset \neg(\phi X)))]$                                                                                      | A1, D1<br>A1, A2, D1, A3, A4, D2, D3, A5                                              | KB<br>KB      | THM<br>THM        | 16.5/—<br>12.8/15.1   | 0.0/0.0<br>0.0/5.4     | —/—<br>—/—            |
| MT  | $[\forall X_{\mu} \cdot \forall Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \supset (g_{\mu \rightarrow \sigma} Y \supset X \equiv Y))]$                                                                                                                                      | D1, FG<br>A1, A2, D1, A3, A4, D2, D3, A5                                              | KB<br>KB      | THM<br>THM        | —/—<br>—/—            | 0.0/3.3<br>—/—         | —/—<br>—/—            |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                  | A1, A2, D1, A3, A4, D2, D3, A5                                                        | KB            | SAT               | —/—                   | —/—                    | 7.3/7.4               |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \forall \psi_{\mu \rightarrow \sigma} (\psi X \supset \diamond \forall Y_{\mu} \cdot (\phi Y \supset \psi Y))$                  | A1( $\supset$ ), A2, D2', D3, A5                                                      | KB            | UNS               | 7.5/7.8               | —/—                    | —/—                   |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                  | A1, A2, D1, A3, A4, D2', D3, A5                                                       | KB            | UNS               | —/—                   | —/—                    | —/—                   |

## Consistency check: Gödel vs. Scott

- ▶ Scott's assumptions are consistent; shown by Nitpick
- ▶ Gödel's assumptions are **inconsistent**; shown by LEO-II (**new philosophical result?**)



# Main Findings

|     | HOL encoding                                                                                                                                                                                                                                            | dependencies                     | logic | status | LEO-II<br>const/vary | Satallax<br>const/vary | Nitpick<br>const/vary |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------|-------|--------|----------------------|------------------------|-----------------------|
| A1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$                                                                                |                                  |       |        |                      |                        |                       |
| A2  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \dot{\forall}\psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} X_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$             |                                  |       |        |                      |                        |                       |
| T1  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\exists} X_{\mu} \cdot \phi X]$                                                                                                    | A1( $\supset$ ), A2              | K     | THM    | 0.1/0.1              | 0.0/0.0                | —/—                   |
|     |                                                                                                                                                                                                                                                         | A1, A2                           | K     | THM    | 0.1/0.1              | 0.0/5.2                | —/—                   |
| D1  | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi$                                                                                                    |                                  |       |        |                      |                        |                       |
| A3  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                          |                                  |       |        |                      |                        |                       |
| C   | $[\dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                            |                                  |       |        |                      |                        |                       |
| A4  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\exists} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi]$                                                                    |                                  |       |        |                      |                        |                       |
| D2  | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda$                                                                                                                                           |                                  |       |        |                      |                        |                       |
| T2  | $[\dot{\forall} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu} X)]$                                                                                                                    |                                  |       |        |                      |                        |                       |
| D3  | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall}\phi_{\mu \rightarrow \sigma} (e$                                                                                                                                              |                                  |       |        |                      |                        |                       |
| A5  | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                  |                                  |       |        |                      |                        |                       |
| T7  | $[\dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                            | D1, C, T2, D3, A5                | K     | CSA    | —/—                  | —/—                    | 3.8/6.2               |
|     |                                                                                                                                                                                                                                                         | A1, A2, D1, A3, A4, D2, D3, A5   | K     | CSA    | —/—                  | —/—                    | 8.2/7.5               |
|     |                                                                                                                                                                                                                                                         | D1, C, T2, D3, A5                | KB    | THM    | 0.0/0.1              | 0.1/5.3                | —/—                   |
|     |                                                                                                                                                                                                                                                         | A1, A2, D1, A3, A4, D2, D3, A5   | KB    | THM    | —/—                  | —/—                    | —/—                   |
| MC  | $[s_{\sigma} \dot{\supset} \dot{\exists} s_{\sigma}]$                                                                                                                                                                                                   | D2, T2, T3                       | KB    | THM    | 17.9/—               | 3.3/3.2                | —/—                   |
|     |                                                                                                                                                                                                                                                         | A1, A2, D1, A3, A4, D2, D3, A5   | KB    | THM    | —/—                  | —/—                    | —/—                   |
| FG  | $[\dot{\forall}\phi_{\mu \rightarrow \sigma} \dot{\forall} X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\supset} \dot{\neg}(\phi X)))]$                             | A1, D1                           | KB    | THM    | 16.5/—               | 0.0/0.0                | —/—                   |
|     |                                                                                                                                                                                                                                                         | A1, A2, D1, A3, A4, D2, D3, A5   | KB    | THM    | 12.8/15.1            | 0.0/5.4                | —/—                   |
| MT  | $[\dot{\forall} X_{\mu} \cdot \dot{\forall} Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} X \dot{\supset} Y))]$                                                                                 | D1, FG                           | KB    | THM    | —/—                  | 0.0/3.3                | —/—                   |
|     |                                                                                                                                                                                                                                                         | A1, A2, D1, A3, A4, D2, D3, A5   | KB    | THM    | —/—                  | —/—                    | —/—                   |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                            | A1, A2, D1, A3, A4, D2, D3, A5   | KB    | SAT    | —/—                  | —/—                    | 7.3/7.4               |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \dot{\forall}\psi_{\mu \rightarrow \sigma} (\psi X \dot{\supset} \dot{\exists} Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                  |       |        |                      |                        |                       |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                            | A1( $\supset$ ), A2, D2', D3, A5 | KB    | UNS    | 7.5/7.8              | —/—                    | —/—                   |
|     |                                                                                                                                                                                                                                                         | A1, A2, D1, A3, A4, D2', D3, A5  | KB    | UNS    | —/—                  | —/—                    | —/—                   |

## Further Results

- Monotheism holds
- God is flawless

# Main Findings

HOL encoding

A1  $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}$   
A2  $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot \dot{\forall} \psi_{\mu \rightarrow \sigma} \cdot (p_{(\mu \rightarrow \sigma) \rightarrow \sigma}$   
T1  $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond} \dot{\exists}$   
D1  $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma}$   
A3  $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$   
C  $[\dot{\diamond} \dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$   
A4  $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond} p_{(\mu \rightarrow \sigma) \rightarrow \sigma}$   
D2  $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda$   
T2  $[\dot{\forall} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} X \dot{\supset} \dot{\diamond} \dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X)]$   
E3  $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot (e$   
A5  $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$   
T3  $[\dot{\diamond} \dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$

## Modal Collapse (Sobel)

$$\forall \phi (\phi \supset \Box \phi)$$

- ▶ proved by LEO-II and Satallax
- ▶ for possibilist and actualist quantification (ind.)

## Main critique on Gödel's ontological proof:

- ▶ there are no contingent truths
- ▶ everything is determined / no free will

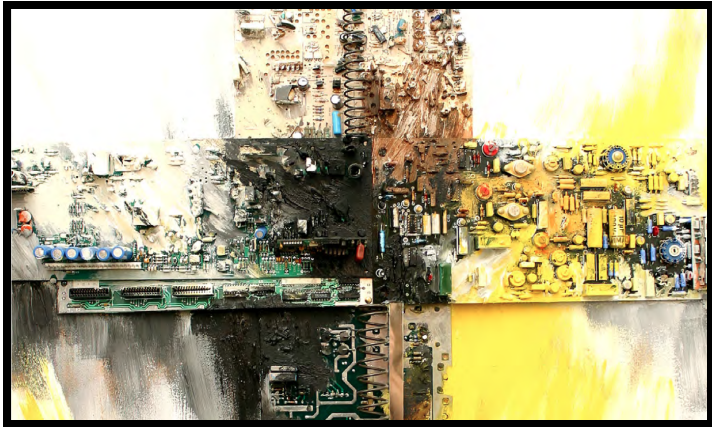
|     |                                                                                                                                                                                                                                                                                                  |                                   |    |     |           |         |         |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------|----|-----|-----------|---------|---------|
| MC  | $[s_{\sigma} \dot{\supset} \dot{\diamond} s_{\sigma}]$                                                                                                                                                                                                                                           | D2, T2, T3                        | KB | THM | 17.9/—    | 3.3/3.2 | —/—     |
|     |                                                                                                                                                                                                                                                                                                  | A1, A2, D1, A3, A4, D2, D3, A5    | KB | THM | —/—       | —/—     | —/—     |
| FG  | $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot \dot{\forall} X_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (\dot{\neg} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\supset} \dot{\neg} (\phi X)))]$                                                             | A1, D1                            | KB | THM | 16.5/—    | 0.0/0.0 | —/—     |
|     |                                                                                                                                                                                                                                                                                                  | A1, A2, D1, A3, A4, D2, D3, A5    | KB | THM | 12.8/15.1 | 0.0/5.4 | —/—     |
| MT  | $[\dot{\forall} X_{\mu} \cdot \dot{\forall} Y_{\mu} \cdot (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} X \dot{\equiv} Y))]$                                                                                                                           | D1, FG                            | KB | THM | —/—       | 0.0/3.3 | —/—     |
|     |                                                                                                                                                                                                                                                                                                  | A1, A2, D1, A3, A4, D2, D3, A5    | KB | THM | —/—       | —/—     | —/—     |
| CO  | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                     | A1, A2, D1, A3, A4, D2, D3, A5    | KB | SAT | —/—       | —/—     | 7.3/7.4 |
| D2' | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \dot{\forall} \psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\diamond} \dot{\forall} Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ | A1 ( $\supset$ ), A2, D2', D3, A5 | KB | UNS | 7.5/7.8   | —/—     | —/—     |
| CO' | $\emptyset$ (no goal, check for consistency)                                                                                                                                                                                                                                                     | A1, A2, D1, A3, A4, D2', D3, A5   | KB | UNS | —/—       | —/—     | —/—     |

# Main Findings

|    | HOL encoding                                                                                                                                                                                                                                                                                               | dependencies                                                                                               | logic              | status                   | LEO-II<br>const/vary         | Satallax<br>const/vary       | Nitpick<br>const/vary            |
|----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|--------------------|--------------------------|------------------------------|------------------------------|----------------------------------|
| A1 | $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_{\mu} \cdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$                                                                                                                            |                                                                                                            |                    |                          |                              |                              |                                  |
| A2 | $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot \dot{\forall} \psi_{\mu \rightarrow \sigma} \cdot (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\neg} \dot{\forall} X_{\mu} \cdot (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$                                             |                                                                                                            |                    |                          |                              |                              |                                  |
| T1 | $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond} \dot{\exists} X_{\mu} \cdot \phi X]$                                                                                                                                 | A1( $\supset$ ), A2<br>A1, A2                                                                              | K<br>K             | THM<br>THM               | 0.1/0.1<br>0.1/0.1           | 0.0/0.0<br>0.0/5.2           | —/—<br>—/—                       |
| D1 | $g_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$                                                                                                                           |                                                                                                            |                    |                          |                              |                              |                                  |
| A3 | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                             |                                                                                                            |                    |                          |                              |                              |                                  |
| C  | $[\dot{\diamond} \dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                                | T1, D1, A3<br>A1, A2, D1, A3                                                                               | K<br>K             | THM<br>THM               | 0.0/0.0<br>0.0/0.0           | 0.0/0.0<br>5.2/31.3          | —/—<br>—/—                       |
| A4 | $[\dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\diamond} p\phi]$                                                                                                                                                              |                                                                                                            |                    |                          |                              |                              |                                  |
| D2 | $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \cdot \lambda X_{\mu} \cdot \phi X \wedge \dot{\forall} \psi_{\mu \rightarrow \sigma} \cdot (\psi X \dot{\supset} \dot{\neg} \dot{\forall} Y_{\mu} \cdot (\phi Y \dot{\supset} \psi Y))$ |                                                                                                            |                    |                          |                              |                              |                                  |
| T2 | $[\dot{\forall} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$                                                                                                                                                  | A1, D1, A4, D2<br>A1, A2, D1, A3, A4, D2                                                                   | K<br>K             | THM<br>THM               | 19.1/18.3<br>12.9/14.0       | 0.0/0.0<br>0.0/0.0           | —/—<br>—/—                       |
| D3 | $\text{NE}_{\mu \rightarrow \sigma} = \lambda X_{\mu} \cdot \dot{\forall} \phi_{\mu \rightarrow \sigma} \cdot (\text{ess } \phi X \dot{\supset} \dot{\neg} \dot{\exists} Y_{\mu} \cdot \phi Y)$                                                                                                            |                                                                                                            |                    |                          |                              |                              |                                  |
| A5 | $[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$                                                                                                                                                                                                                     |                                                                                                            |                    |                          |                              |                              |                                  |
| T3 | $[\dot{\neg} \dot{\diamond} \dot{\exists} X_{\mu} \cdot g_{\mu \rightarrow \sigma} X]$                                                                                                                                                                                                                     | D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5<br>D1, C, T2, D3, A5<br>A1, A2, D1, A3, A4, D2, D3, A5 | K<br>K<br>KB<br>KB | CSA<br>CSA<br>THM<br>THM | —/—<br>—/—<br>0.0/0.1<br>—/— | —/—<br>—/—<br>0.1/5.3<br>—/— | 3.8/6.2<br>8.2/7.5<br>—/—<br>—/— |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | THM                      | 17.9/—                       | 3.3/3.2                      | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | THM                      | —/—                          | —/—                          | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | THM                      | 16.5/—                       | 0.0/0.0                      | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | THM                      | 12.8/15.1                    | 0.0/5.4                      | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | THM                      | —/—                          | 0.0/3.3                      | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | THM                      | —/—                          | —/—                          | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | SAT                      | —/—                          | —/—                          | 7.3/7.4                          |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | UNS                      | 7.5/7.8                      | —/—                          | —/—                              |
|    |                                                                                                                                                                                                                                                                                                            |                                                                                                            | KB                 | UNS                      | —/—                          | —/—                          | —/—                              |

## Observation

- ▶ good performance of ATPs
- ▶ excellent match between argumentation granularity in papers and the reasoning strength of the ATPs



## Reconstruction of the Inconsistency of Gödel's Axioms

# Inconsistency (Gödel): Proof by LEO-II in KB

```

DemoMaterial — bash — 166x52

@SV8@SV3)=false) | (((pe(^[X0:mu,SX1:si]: sfalse))@SV3)=true)),inference(prim_subst,[status(thm)],l66:[bind(SV11,sthf(^[SV23:mu,SV24:si]: sfalse))]])),
thf(l84,plain,([SV22:(mu>(si>so)),SV3:si,SV8:(mu>(si>so))]: (((SV8@((sk2_SY33@SV3)@(^[X0:mu,SX1:si]: (~ (SV22@SX0@SX1)))@SV8))@((sk1_SY31@(^[X0:mu,SX1:si]: (~
((SV22@SX0@SX1)))@SV8@SV3)=true) | (((peSV8@SV3)=sfalse) | (((pe(^[X0:mu,SX1:si]: (~ (SV22@SX0@SX1)))@SV3)=true))),inference(prim_subst,[status(thm)],l66
:[bind(SV11,sthf(^[SV20:mu,SV21:si]: (~ (SV22@SV20@SV21)))])),
thf(l85,plain,([SV4:si,SV9:(mu>(si>so))]: (((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)=false) | (((peSV9@SV4)= (pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27
@SY28)))@SV4)=false))),inference(fac_restr,[status(thm)],l56l)),
thf(l86,plain,([SV4:si,SV9:(mu>(si>so))]: (((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4)=true) | (((peSV9@SV4)= (pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@
SY30)))@SV4)=false))),inference(fac_restr,[status(thm)],l57l)),
thf(l87,plain,([SV4:si,SV9:(mu>(si>so))]: (((~ ((peSV9@SV4) | ((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)) | (~ ((~ ((peSV9@SV4)) | (~ ((pe(^[SY27:mu,
SY28:si]: (~ (SV9@SY27@SY28)))@SV4)))))=false) | (((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)=false))),inference(extcnf_equal_neg,[status(thm)],l85l)),
thf(l89,plain,([SV4:si,SV9:(mu>(si>so)) | ((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4)) | (~ ((~ ((peSV9@SV4)) | (~ ((pe(^[SY29:mu,
SY30:si]: (~ (SV9@SY29@SY30)))@SV4)))))=false) | (((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4)=true))),inference(extcnf_equal_neg,[status(thm)],l86l)),
thf(l92,plain,([SV4:si,SV9:(mu>(si>so))]: (((~ ((peSV9@SV4) | (~ ((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)))))=false) | (((pe(^[SY27:mu,SY28:si]: (~
((SV9@SY27@SY28)))@SV4)=false))),inference(extcnf_or_neg,[status(thm)],l87l)),
thf(l93,plain,([SV4:si,SV9:(mu>(si>so))]: (((~ ((peSV9@SV4) | ((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4)))))=false) | (((pe(^[SY29:mu,SY30:si]: (~ (SV9@
SY29@SY30)))@SV4)=true))),inference(extcnf_or_neg,[status(thm)],l89l)),
thf(l96,plain,([SV4:si,SV9:(mu>(si>so))]: (((~ ((peSV9@SV4) | (~ ((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)))))=true) | (((pe(^[SY27:mu,SY28:si]: (~ (S
V9@SY27@SY28)))@SV4)=false))),inference(extcnf_not_neg,[status(thm)],l92l)),
thf(l97,plain,([SV4:si,SV9:(mu>(si>so))]: (((peSV9@SV4) | ((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4))=true) | (((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29
@SY30)))@SV4)=true))),inference(extcnf_not_neg,[status(thm)],l93l)),
thf(l100,plain,([SV4:si,SV9:(mu>(si>so))]: (((~ ((peSV9@SV4)=true) | (~ ((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4))=true) | (((pe(^[SY27:mu,SY28:si]: (~
((SV9@SY27@SY28)))@SV4)=false))),inference(extcnf_or_pos,[status(thm)],l96l)),
thf(l101,plain,([SV4:si,SV9:(mu>(si>so))]: (((peSV9@SV4)=true) | ((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4)=true) | (((pe(^[SY29:mu,SY30:si]: (~ (SV
9@SY29@SY30)))@SV4)=false))),inference(extcnf_or_pos,[status(thm)],l97l)),
thf(l123,plain,([SV4:si,SV9:(mu>(si>so))]: (((peSV9@SV4)=false) | (~ ((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4))=true) | (((pe(^[SY27:mu,SY28:si]: (~
((SV9@SY27@SY28)))@SV4)=false))),inference(extcnf_not_pos,[status(thm)],l100l)),
thf(l105,plain,([SV4:si,SV9:(mu>(si>so))]: (((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)=false) | ((peSV9@SV4)=false) | (((pe(^[SY27:mu,SY28:si]: (~ (S
V9@SY27@SY28)))@SV4)=false))),inference(extcnf_not_pos,[status(thm)],l103l)),
thf(l107,plain,([SV8:(mu>(si>so)),SV3:si,SV22:(mu>(si>so))]: (((SV22@((sk2_SY33@SV3)@(^[X0:mu,SX1:si]: (~ (SV22@SX0@SX1)))@SV8))@((sk1_SY31@(^[X0:mu,SX1:si]: (~
((SV22@SX0@SX1)))@SV8@SV3)=true) | (((peSV8@SV3)=false) | (((pe(^[X0:mu,SX1:si]: (~ (SV22@SX0@SX1)))@SV3)=true))),inference(extcnf_not_neg,[status(thm
)],l78l)),
thf(l108,plain,([SV11:(mu>(si>so)),SV3:si,SV15:(mu>(si>so))]: (((SV15@((sk2_SY33@SV3)@SV11)@(^[X0:mu,SX1:si]: (~ (SV15@SX0@SX1)))@((sk1_SY31@SV11)@(^[X0:mu,
SX1:si]: (~ (SV15@SX0@SX1)))@SV3)=false) | (((pe(^[X0:mu,SX1:si]: (~ (SV15@SX0@SX1)))@SV3)=false) | (((peSV11@SV3)=true))),inference(extcnf_not_pos,[statu
s(thm)],l81l)),
thf(l109,plain,([SV4:si,SV9:(mu>(si>so))]: (((pe(^[SY27:mu,SY28:si]: (~ (SV9@SY27@SY28)))@SV4)=false) | ((peSV9@SV4)=false))),inference(sim,[status(thm)],l10
5l)),
thf(l110,plain,([SV4:si,SV9:(mu>(si>so))]: (((peSV9@SV4)=true) | (((pe(^[SY29:mu,SY30:si]: (~ (SV9@SY29@SY30)))@SV4)=true))),inference(sim,[status(thm)],l101
l)),
thf(l111,plain,([SV3:si,SV8:(mu>(si>so))]: (((peSV8@SV3)=false) | ((pe(^[X0:mu,SX1:si]: true))@SV3)=true))),inference(sim,[status(thm)],l76l)),
thf(l112,plain,([SV11:(mu>(si>so)),SV3:si]: (((pe(^[X0:mu,SX1:si]: false))@SV3)=false) | ((peSV11@SV3)=true))),inference(sim,[status(thm)],l80l)),
thf(l113,plain,([false)=true]),inference(fo_atp_e,[status(thm)],l25,l12,l11,l10,l89,l88,l87,l84,l83,l82,l75,l74,l73,l72,l71,l70,l69,l68,l67,l66,l65,l62,l57,l56,l54,l29)),
thf(l114,plain,([false]),inference(solved_all_splits,[status(thm)],l113l)),
% SZ5 output end,OffRefutation

***** End of derivation protocol *****
no. of clauses in derivation: 97 *****
clause counter: 113 *****

% SZ5 status Unsatisfiable for ConsistencyWithoutFirstConjunctinD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibEQ:true,rAndEQ:true,use_choice:true,use_extuni:true,use
extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,transl
ation:fof_full)
ontoleo:DemoMaterial cbenzmueller's []
```

## Inconsistency (Gödel): Informal Argument

**Def. D2\***

$$\varphi \text{ ess } x \leftrightarrow \cancel{\varphi(x)} \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\varphi(y) \rightarrow \psi(y)))$$

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►

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► by L1

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► by def. of  $\emptyset$

$$\Diamond \exists x [\top \rightarrow \Box \perp]$$

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**Inconsistency**



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$\perp$

by reflexivity

**Inconsistency**

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$$\Diamond \exists x [\top \rightarrow \Box \exists y [\emptyset(y)]]$$

► by def. of  $\emptyset$

$$\Diamond \exists x [\top \rightarrow \Box \perp]$$

$\perp$

by reflexivity  
or by symmetry

**Inconsistency**

# Inconsistency (Gödel): Informal Argument

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►

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► by L1

$$\Diamond \exists x [\top \rightarrow \Box \exists y [\emptyset(y)]]$$

► by def. of  $\emptyset$

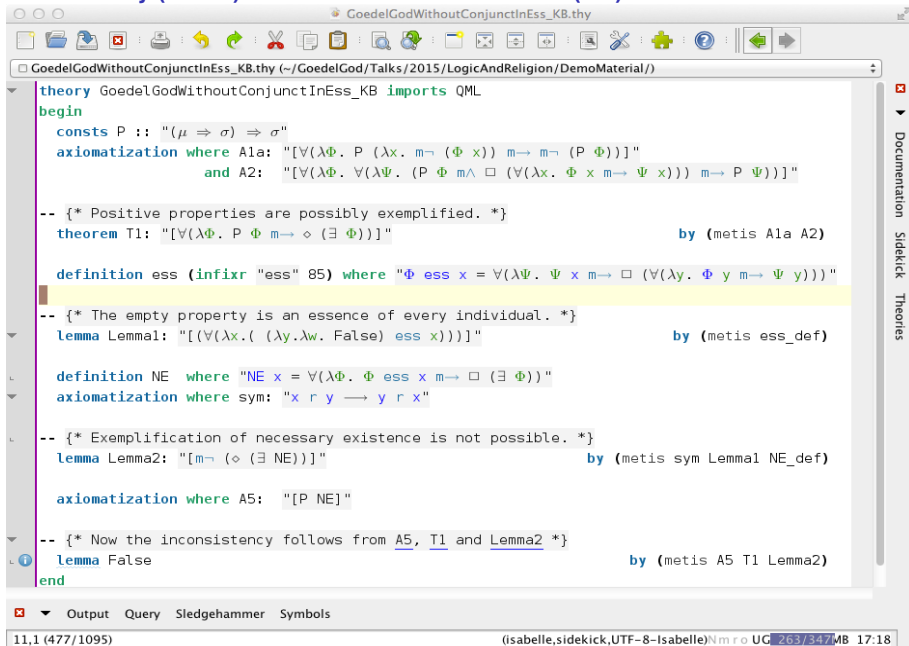
$$\Diamond \exists x [\top \rightarrow \Box \perp]$$

$\perp$

by reflexivity  
or by symmetry  
or by nothing!

**Inconsistency**

# Inconsistency (Gödel): Verification in Isabelle/HOL (KB)



GoedelGodWithoutConjunctInEss\_KB.thy

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
  consts P :: "( $\mu \Rightarrow \sigma$ )  $\Rightarrow \sigma$ "
  axiomatization where A1a: "[ $\forall(\lambda\Phi. P (\lambda x. \neg (\Phi x))) \rightarrow \neg (P \Phi)$ ]"
    and A2: "[ $\forall(\lambda\Phi. \forall(\lambda\Psi. (P \Phi \wedge \Box (\forall(\lambda x. \Phi x \rightarrow \Psi x))) \rightarrow P \Psi))$ ]"

  -- {* Positive properties are possibly exemplified. *}
  theorem T1: "[ $\forall(\lambda\Phi. P \Phi \rightarrow \Diamond (\exists \Phi))$ ]" by (metis A1a A2)

  definition ess (infixr "ess" 85) where " $\Phi$  ess x =  $\forall(\lambda\Psi. \Psi x \rightarrow \Box (\forall(\lambda y. \Phi y \rightarrow \Psi y)))$ "

  -- {* The empty property is an essence of every individual. *}
  lemma Lemmal: "[ $(\forall(\lambda x. (\lambda y. \text{False}) \text{ess } x))$ ]" by (metis ess_def)

  definition NE where " $\text{NE } x = \forall(\lambda\Phi. \Phi \text{ess } x \rightarrow \Box (\exists \Phi))$ "
  axiomatization where sym: " $x \text{ r } y \longrightarrow y \text{ r } x$ "

  -- {* Exemplification of necessary existence is not possible. *}
  lemma Lemma2: "[ $\neg (\Diamond (\exists \text{NE}))$ ]" by (metis sym Lemmal NE_def)

  axiomatization where A5: "[ $P \text{NE}$ ]"

  -- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
  lemma False by (metis A5 T1 Lemma2)
end
```

11,1 (477/1095) (isabelle,sidekick,UTF-8-Isabelle)Nr o UG\_263/347MB 17:18

# Inconsistency (Gödel): Verification in Isabelle/HOL (K)

GoedelGodWithoutConjunctInEss\_K.thy

GoedelGodWithoutConjunctInEss\_K.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: "( $\mu \Rightarrow \sigma$ )  $\Rightarrow \sigma$ "
  definition ess (infixr "ess" 85) where " $\Phi$  ess x =  $\forall(\lambda\Psi. \Psi\ x\ m\rightarrow \Box (\forall(\lambda y. \Phi\ y\ m\rightarrow \Psi\ y)))$ "
  definition NE where " $NE\ x = \forall(\lambda\Phi. \Phi\ ess\ x\ m\rightarrow \Box (\exists\ \Phi))$ "
  axiomatization where A1a: " $[\forall(\lambda\Phi. P\ (\lambda x. m\rightarrow (\Phi\ x))\ m\rightarrow m\rightarrow (P\ \Phi))]$ "
    and A2: " $[\forall(\lambda\Phi. \forall(\lambda\Psi. (P\ \Phi\ m\wedge \Box (\forall(\lambda x. \Phi\ x\ m\rightarrow \Psi\ x)))\ m\rightarrow P\ \Psi))]$ "

  -- {* Positive properties are possibly exemplified. *}
  theorem T1: " $[\forall(\lambda\Phi. P\ \Phi\ m\rightarrow \Diamond (\exists\ \Phi))]$ " by (metis A1a A2)

  -- {* The empty property is an essence of every individual. *}
  lemma Lemmal: " $[(\forall(\lambda x. ((\lambda y. \lambda w. False)\ ess\ x)))]$ " by (metis ess_def)

  axiomatization where A5: "[P NE]"

  -- {* Now the inconsistency follows from A5, Lemmal, NE_def and T1 *}
  lemma False
  -- {* sledgehammer [remote_leo2] *}
  by (metis A5 Lemmal NE_def T1)

end
```

Output Query Sledgehammer Symbols

21,7 (980/982) (isabelle,sidekick,UTF-8-Isabelle)Nm r o UG 302/343M8 11:37

Documentation Sidekick Theories

## Summary: Results of Experiments

| Gödel's version | K        |          | KB       |          | S5       |          |
|-----------------|----------|----------|----------|----------|----------|----------|
|                 | constant | varying  | constant | varying  | constant | varying  |
| Consistency     | ×        | ×        | ×        | ×        | ×        | ×        |
| T1              | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |
| C               | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |
| T2              | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |
| T3              | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |
| Flawless God    | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |
| Monotheism      | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |
| Modal Collapse  | obsolete | obsolete | obsolete | obsolete | obsolete | obsolete |

### Further logic details

- ▶ Henkin semantics
- ▶ full comprehension
- ▶ rigid constant symbols

### Question:

**Has this inconsistency been reported before?**

If not, then LEO-II deserves (part of) the credit!

## Summary: Results of Experiments

| Scott's version | K        |         | KB       |         | S5       |         |
|-----------------|----------|---------|----------|---------|----------|---------|
|                 | constant | varying | constant | varying | constant | varying |
| Consistency     | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |
| T1              | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |
| C               | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |
| T2              | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |
| T3              | ×        | ×       | ✓        | ✓       | ✓        | ✓       |
| Flawless God    | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |
| Monotheism      | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |
| Modal Collapse  | ✓        | ✓       | ✓        | ✓       | ✓        | ✓       |

### Further logic details

- ▶ Henkin semantics
- ▶ full comprehension
- ▶ rigid constant symbols

# Avoiding the Modal Collapse: Recent Variants

## SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

## Gödel's Ontological Proof Revisited \*

C. Anthony Anderson and Michael Gettings  
University of California, Santa Barbara  
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [6] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

PĚTR HÁJEK

## A New Small Emendation of Gödel's Ontological Proof

**Keywords:** Ontological proof, Gödel, modal logic, comprehension, positive properties.

### 1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needless to say, formal modal logic has found several

## Magari and others on Gödel's ontological proof

Petr Hájek  
Institute of Computer Science, Academy of Sciences  
182 07 Prague, Czech Republic  
e-mail: hajek@uivt.cas.cz

### 1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

## Der Mathematiker und die Frage der Existenz Gottes (betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,  
sondern glauben, daß ein Gott sei.  
(Kant, Metaphik)

### 1. Einführung

Gödel's zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödel'schen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödel's Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung“<sup>1</sup> zu schreiben, ohne aus ihr einen

## Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula  $P(F)$  stand for "the property  $F$  is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:



# Avoiding the Modal Collapse: Some Emendations

## SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

## Gödel's Ontological Proof Revisited \*

C. Anthony Anderson and Michael Gettings

University of Cambridge, Santa Barbara  
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [6] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

Petr Hájek

## A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

### 1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [10], but it became known in 1970 when Gödel showed the proof to Daniel Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof [2, 7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needless to say, formal modal logic has become a universal

## Magari and others on Gödel's ontological proof

Petr Hájek

Institute of Computer Science, Academy of Sciences  
182 07 Prague, Czech Republic  
e-mail: hajek@uivt.cas.cz

### 1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

## Der Mathematiker und die Frage der Existenz Gottes (betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,  
sondern glauben, daß ein Gott sei.  
(Kant, Metaph.)

### 1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödel'schen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Fein von Professor Gert Müller (Heidelberg, Januar 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung“<sup>1</sup> zu schreiben, ohne aus ihr einen

## Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula  $P(F)$  stand for "the property  $F$  is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

## Computer-supported Clarification of Controversy

Higher-Order Modal Logics  
(Lecture 4 - Part 2)

Controversial Variants of the Ontological Argument

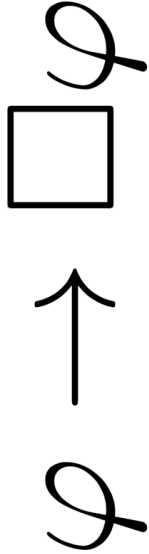
Christoph Benz Müller

Bruno Woltzenlogel Paleo

(joint work with Leon Weber)

# Sobel's Modal Collapse

1987



# Magari's Claim

1988

## 4. *Indebolimento dei primi tre assiomi*

Come si è visto i primi tre assiomi e la definizione di  $G$  sono sufficienti a dare il teorema principale, ma chiaramente non era questa l'intenzione di A: A intendeva assicurarsi il teorema  $1$ ,  $\exists m \exists x G_m x$ , e ricavarne, usando anche gli altri assiomi e in particolare i concetti di essenza e di esistenza necessaria, il teorema principale. Tutto sommato sembra plausibile immaginare di scegliere i primi tre assiomi, conservando l'assioma definito  $Dr$ . In questo modo, vedendo le cose in un mo-

"A4 and A5 are **superfluous**"

# Magari's Claim

1988

"A4 and A5 are **superfluous**"

A4 Positive properties are necessarily positive:

$$\forall \varphi [P(\varphi) \rightarrow \Box P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\varphi(y) \rightarrow \psi(y)))$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \equiv \forall \varphi [\varphi \text{ ess } x \rightarrow \Box \exists y \varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

# Anderson's Emendation

1990

A:A1 If a property is positive, its negation is not positive:

$$\forall\varphi[P(\varphi) \rightarrow \neg P(\neg\varphi)]$$

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

A:D2 An *essence* of an individual is a property that necessarily implies those and only those properties that the individual has necessarily:

$$\varphi \text{ ess}_A x \equiv \forall\psi[\Box\psi(x) \leftrightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y))]$$

# Hájek's Claims 1996 - 2001

are to be assumed? And (3), what *modal axioms* are to be assumed? Sobel [S] and Polívka (cf.[H]) observed that assuming full comprehension leads to *collapse of modalities*, i.e.  $(\forall X)(\forall x)(X(x) \equiv \Box X(x))$  become provable, which means that the system trivializes. Magari [M] claims that the first three Gödel's axioms  $(A1) - (A3)$  already imply the main theorem; we shall analyze his claim in Section 2 and show that his proof implicitly uses a too strong equality axiom (without which the claim is not true, as shown in [H]). Anderson [A] presented an emended system admitting full comprehension and not suffering by collapse of modalities; I proved in [H] that for Anderson's system Magari's claim becomes true; the analogs of Gödel's  $A1 - A3$  prove necessary existence of a godlike individual (in the modified sense), but

**5.3 Theorem:**  $AO \vdash (A4^*), (C4^*), (A5^*)$ .

*Beweis.*  $(A4^*)$  ist  $P(Y) \rightarrow \Box P(Y)$ . Sei  $G^*(g)$ ; dann ist  $P(Y) \equiv \Box Y(g)$ , also  $\Box P(Y) \equiv \Box \Box Y(g) \equiv \Box Y(g)$ , und daher  $P(Y) \equiv \Box P(Y)$ .

"Magari is wrong about **superfluosness** of  $A4$  and  $A5$ "  
"But  $A4$  and  $A5$  are **redundant** in Anderson's Theory"



## Rebuttal of Anderson and Gettings 1996

<sup>1</sup> Petr Hájek [13] has argued that Anderson's version of the argument has superfluous premisses, but the truth of this claim depends on the details of the underlying second-order modal logic adopted. In the context of the (quite reasonable) version of that logic formulated by Nino Cocchiarella [2] and cited by

Anderson as the underlying logic, Hájek's point does *not* hold. It does indeed follow from the diminished set of premisses that: if THERE IS a Godlike being, then necessarily THERE IS such a being. From this one can easily deduce that THERE IS a Godlike being, as Hajek observes. But the sense of the quantifier indicated by the capitalized phrase here (i.e., as formalized in Cocchiarella's logic) is that of "possible existence" or "subsistence". To express actual existence requires a separate quantifier. So without using the other premisses, nothing yet follows about the actual existence of a Godlike being.

"Hájek's claim assumes **constant domain semantics**.

But Anderson assumed **varying domain semantics**.

Hájek's claim is wrong under the correct semantics."

A simplification of the emendation is defined



## Varying Domains

`text` {\* Technically, varying domains are encoded with the help of an existence relation expressing which individuals actually exist in each world.\*}

`consts eiw :: " $\mu \Rightarrow i \Rightarrow \text{bool}$ "`

`axiomatization where` nonempty: " $\forall w. \exists x. \text{eiw } x \ w$ "

`text` {\* Actualistic quantifiers are quantifiers guarded by the existence relation.\*}

`abbreviation mforall :: " $(\mu \Rightarrow \sigma) \Rightarrow \sigma$ "` (" $\forall e$ ")

`where` " $\forall e \ \Phi \equiv (\lambda w. \forall x. (\text{eiw } x \ w) \longrightarrow (\Phi \ x \ w))$ "

`abbreviation mexistse :: " $(\mu \Rightarrow \sigma) \Rightarrow \sigma$ "` (" $\exists e$ ")

`where` " $\exists e \ \Phi \equiv (\lambda w. \exists x. (\text{eiw } x \ w) \wedge \Phi \ x \ w)$ "

# Anderson's Long Footnote

Footnote 1990

14. One should consult Cocchiarella's article for a detailed and precise account of his semantics for second-order modal logic. But, at least for those having some familiarity with the usual "possible world" approach, the following outline of the semantics may make possible the present argument. We are to imagine that a set of  $n$  possible worlds,  $\{w_1, \dots, w_n\}$ , exist. The set of all possible individuals is the union of the individuals existing in the world, a set  $B$  (the "possible individuals"; the set of actual individuals must be a subset of this set) and an interpretation function  $R$  which assigns appropriate extensions to the constants of the language—possible individuals to individual constants, sets of possible individuals to one-place predicates and so on. It is worth emphasizing that the denotation of a constant (as given by  $R$ ) at a world and the extension of a predicate (also given by  $R$ ) at a world need not consist of individuals which are "actual" at the world. And Cocchiarella's logic has two kinds of quantifiers: what we might call "substantial quantifiers" which (in the semantics) are construed as ranging over all possible individuals (the set  $B$ ), and "e-quantifiers," existence quantifiers, which are interpreted at a world as ranging only over the entities which are "actual" at the world. The logic is interpreted by giving a model to a system—a collection of two sets, one for possible individuals and one for actual individuals.

That the possible individuals of each  $L$ -model be the same as those of any other (the possible individuals are the same no matter what world you are in). (Technically the  $L$ -models are "indexed"; the set  $I$  is the domain of a function which yields an  $L$ -model for each  $i$  belonging to  $I$ . And the existing individuals of all the worlds, taken together, must be a subset of the set of possible individuals. A singular attribute is as usual a function which takes as arguments possible worlds and yields as value in each case a set of possible individuals. And we may take the one-place second-order variables as ranging over all the individuals which correspond to a given world system (Cocchiarella has two kinds of second-order variables, but only one is relevant to our present concern). The  $n$ -ary attributes are defined analogously, ignoring for simplicity some not immediately relevant complications of Cocchiarella's construction; our default world system may be taken to be a world with  $n$  possible individuals,  $\{a_1, \dots, a_n\}$ , and  $n$  actual individuals,  $\{a_1, \dots, a_n\}$ .  $R_1, R_2$  and  $A_1, A_2 = \{a_1, \dots, a_n\}$ . The first set listed in each case contains the actual individuals of the world, the second contains the possible individuals, and  $R_3$  assigns extensions (from the second set) at the world to the constants and predicates of the language. The singularly attributes of the world system are therefore all the functions whose ranges consist of just the two worlds and whose values are in each case set of possible individuals. The  $n$ -ary attributes ( $n > 1$ ) of the world system are defined analogously.

## EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

301

gously (although we do not actually care about them for the present purpose). Now use the notation  $\Phi_{a^*}$  to stand for the attribute that picks out singleton  $a$  at the world  $1$  (corresponding to  $A_1$ ) and picks out the set of all possible individuals  $x = \{a, b\}$  at world  $2$  (corresponding to  $A_2$ ). In this way, we can define the other first-order attributes that exist in this world system. Then take  $R_1(A_1) = R_2(A_2) = \Phi_{a^*}$ . It is easy to check that the only positive attributes (those whose negations entail  $\Phi_{a^*}$ ) are  $\Phi_{a^*}$ ,  $\Phi_{b^*}$ , and  $\Phi_{a,b}$ , and that the first of these is the attribute corresponding to "G" and "NE" and is the unique essence of  $a$ . Note well that the appropriate translation of the argument into Cocchiarella's notation will take the existential quantifier in the definition of necessary existence to be an  $e$ -quantifier (an "existence," rather than "substantial," quantifier) and so too the quantifier in the conclusion of the argument. All others individual quantifiers may be taken to be substantial and hence to range over all possibilities. It is a purely combinatorial task to show that the modified Gödelian axioms all come out true in this model and that there is no model collapse. And not everything is a necessary statement. Cocchiarella proves that his axioms for second-order  $S5$  are complete in the "Henkin-sense." Given the definition of "positive property," one can completely formalize the present ontological proof in that system. Thus, there is at least one formalization of the present proof using a reasonably adequate logic in which no model collapse is demonstrable.

17 footnotes

4 pages of footnotes

in an 8-page paper

a strange variant with mixed quantifiers is indicated

# Hájek's Reaction to the Rebuttal 2002

DEFINITION 8.  $\mathcal{AO}$  is the system of axioms (A1)–(A5) and definitions of  $\mathbf{H}$ ,  $\mathbf{Ess}$ ,  $\mathbf{NE}$  just presented (Anderson ontological).  $\mathcal{AO}_0$  is the subsystem of axioms (A1)–(A3) plus the definition of  $\mathbf{H}$ .

In [9, 8] I showed that assuming our logic as above, (A4)–(A5) are redundant since (A1)–(A3) prove both  $\Box(\exists x)\mathbf{H}(x)$  and also (A4) and (A5) (using the definitions of  $\mathbf{Ess}$ ,  $\mathbf{NE}$ ). In other words,  $\mathcal{AO}$  and  $\mathcal{AO}_0$  are equivalent in the logic  $\mathcal{L}_{\text{ont}}$ . But in [2] the authors stress that Anderson's proof works in a finer logic in which all the axioms are needed. We shall discuss this in Section 4; the next section is devoted to the analysis and emendation of  $\mathcal{AO}_0$  in our present logic.

# Hájek's Reaction to the Rebuttal 2002

Our main results are the following:

1. In the logic  $\mathcal{L}_{\text{ont}}$  with fixed domains, the system  $\mathcal{AO}'_0$  of axioms

$$(A12) \quad (P(X) \& \Box(X \subseteq Y)) \rightarrow \neg P(\neg Y)$$

$$(\text{def } 1) \quad H(x) \equiv (\forall Y)(\Box Y(x) \equiv (\exists Z)(P(Z) \& \Box(Z \subseteq Y)))$$

$$(A3) \quad P(H)$$

proves  $\Box(\exists x)H(x)$ . (Modified fragment of Anderson.)

2. In the logic  $\mathcal{L}_{\text{ont}, E}$  with the existence predicate  $E$ , the system  $\mathcal{AOE}'_0$  of axioms

$$(A12)^E \quad (P(X) \& \Box(X \subseteq^E Y)) \rightarrow \neg P(\neg Y)$$

$$(\text{def } 1) \quad H(x) \equiv (\forall Y)(\Box Y(x) \equiv (\exists Z)(P(Z) \& \Box(Z \subseteq^E Y)))$$

$$(A3)^E \quad P(H \& E).$$

proves  $\Box(\exists^E x)H(x)$ . (Modified fragment of Anderson with existence.)

# Hájek's Reaction to the Rebuttal 2002

DEFINITION 13.  $\mathcal{AOE}''$  in the following modification of  $\mathcal{AOE}'$  :

- axioms  $(A12)^E$ ,  $(A3)$ ,
- definition of  $H$  as in  $\mathcal{AOE}'$ ,
- definition  $P^\#(Y) \equiv (\exists Z)(P(Z) \ \& \ \Box(Z \subseteq Y))$   
(thus the definition of  $H$  can be written as

$$H(u) \equiv (\forall Y)(\Box Y(u) \equiv P^\#(Y))),$$

$$(A4)^\# \quad P(Y) \rightarrow \Box P^\#(Y)$$

$$(A5)^\# \quad P^\#(NE).$$

# Bjørddal's Proof

1998

B:D1 A property is positive iff it is necessarily possessed by every God-like being.

$$P_B(\phi) \equiv \Box \forall x (G_B(x) \rightarrow \phi(x))$$

B:D2 a *maximal composite* of an individual's positive properties is a positive property possessed by the individual and necessarily implying every positive property possessed by the individual.

$$\text{MCP}(\phi, x) \equiv (\phi(x) \wedge P_B(\phi)) \wedge \forall \psi ((\psi(x) \wedge P_B(\psi)) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$$

B:D3 *Necessary existence* of an individual is the necessary exemplification of all its maximal composites.

$$\text{NE}_B(x) \equiv \forall \phi (\text{MCP}(\phi, x) \rightarrow \Box \exists y \phi(y))$$

A:A1' If a property is positive, its negation is not positive:

$$\forall \varphi [P_B(\varphi) \rightarrow \neg P_B(\neg \varphi)]$$

A5' Necessary existence is a positive property.

$$P_B(\text{NE}_B)$$

# Bjørddal's Proof

1998

A5'

I have later been able to improve upon the result reported here. By making use of a result by Petr Hájek,<sup>3</sup> which he made we aware of at the Liblice-conference, and presupposing certain recursive definition-clauses for *divine* (positive) and *godly being*, we may show that even Ax. 2 is eliminable if we presuppose a reasonable second order comprehension principle for the predicate *godly being*. And so it in fact turns out that, modulo the (reasonable, I think) logical apparatus presupposed, only Ax. 1 is needed in order to derive the theistic conclusion. I hope to be able to publish this improved result, alongside with certain remarks, in a future paper.

# Our Goal

Follow Leibniz's dictum!

Controversy between  
Anderson and Hájek.

Who is right?

Let the "calculus ratiocinator"  
compute and decide!





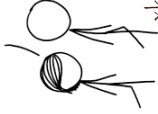
# What we did...

Computer-Assisted Analysis of  
11 variants of  
Gödel's Ontological Argument

Discovery of flawed claims

Strengthening of correct claims

...BUT WOULDN'T A GOD  
WHO COULD FIND A FLAW IN  
THE ONTOLOGICAL ARGUMENT  
BE EVEN *GREATER*?



xkcd.com

## Redundancy of A4 and A5 in Anderson's Emendation in "Isabelle"

```
Anderson_var.thy
Anderson_var.thy (~/Dropbox/Psiquis/Projects/CoqdeCoq/Formalizations/Isabelle/Anderson)
theorem A4: "[ $\forall(\lambda\phi. P \phi \rightarrow \Box (P \phi))$ ]"
(* sledgehammer [remote_satallax remote_leo2] *)
by (metis A3 G_def sym trans T3)

definition ess :: " $\mu \Rightarrow \sigma \Rightarrow \mu \Rightarrow \sigma$ " where
"ess = ( $\lambda\phi. \lambda x. ((\forall(\lambda\psi. ((\Box (\psi x)) \models \Box (\forall(\lambda y. \phi y \rightarrow \psi y))))))$ )"

definition NE :: " $\mu \Rightarrow \sigma$ " where
"NE = ( $\lambda x. \forall(\lambda\phi. \text{ess } \phi x \rightarrow (\Box (\exists(\lambda y. \phi y))))$ )"

theorem A5: "[P NE]"
by (metis A2 A3 ess_def NE_def)

subsection {* Consistency again (now with sym and trans) *}
subsection {* Immunity to Modal Collapse *}

lemma MC: "[ $\forall(\lambda\phi. (\phi \rightarrow (\Box \phi)))$ ]"
nitpick [user_axioms] oops
```

66,34 (17297/2570) | Isabelle, Sidekick, UTT-8-Isabelle | 1 MB 8:17 PM

# Computer-Assisted Analysis

| Proof                 | D1'     | A:A1' | A2' | A3' | A4      | A4'     | H:A4 | A5  | A5' | H:A5 | T3     | T3' | MC      |
|-----------------------|---------|-------|-----|-----|---------|---------|------|-----|-----|------|--------|-----|---------|
| Scott (const)         | -       | -     | -   | -   | N/I     | -       | -    | N/I | -   | -    | P      | -   | P       |
| Scott (var)           | -       | -     | -   | -   | N/I     | -       | -    | N/I | -   | -    | P      | -   | P       |
| Anderson (const)      | -       | -     | -   | -   | R (KB)  | -       | -    | R   | -   | -    | P      | -   | CS      |
| Anderson (var)        | -       | -     | -   | -   | R (KB)  | -       | -    | R   | -   | -    | P      | -   | CS      |
| Anderson (mix)        | -       | -     | -   | -   | R (K4B) | -       | -    | I   | -   | -    | CS     | -   | CS      |
| Hájek AOE' (var)      | -       | -     | CS  | -   | S/I     | -       | -    | S/I | -   | -    | P (KB) | -   | CS (S5) |
| Hájek AOE''_0 (var)   | -       | -     | -   | CS  | R       | -       | -    | S/U | -   | -    | P (KB) | -   | CS (S5) |
| Hájek AOE''' (var)    | -       | -     | -   | -   | R       | -       | -    | S/I | -   | -    | P (KB) | -   | CS (S5) |
| Anderson (simp) (var) | -       | R     | R   | -   | -       | R (K4B) | -    | -   | -   | -    | -      | -   | CS      |
| Bjærdal (const)       | R (K4)  | -     | R   | R   | -       | R (KT)  | -    | N/I | -   | -    | P (KB) | -   | CS      |
| Bjærdal (var)         | CS (S5) | -     | R   | R   | -       | R (KT)  | -    | N/I | -   | -    | P (KB) | -   | CS      |

P = provable; CS = counter-satisfiable;

R = redundant; I = independent;

S = superfluous; N = not superfluous

# Disclaimer

The analyzed papers had ambiguities and lacked details.

We did the best we could, but our formalizations may still differ from what the authors intended.

In contrast to the informal papers, the formal automatic proofs are as precise as currently possible.

# Conclusion

Higher-Order Automated Reasoners  
are already sufficiently  
**effective** and **user-friendly**  
for assisting us in the  
development of philosophical theories.

Leibniz's dream is a reality today

# References

◆ [github.com/FormalTheology/GoedelGod](https://github.com/FormalTheology/GoedelGod)

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Formalizations of Gödel's Ontological Proof of God's Existence — Edit

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update of paper

Merge branch 'ND'

final submission

added link

add minue and mislase

Gitignore \*.fim files

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2 months ago

2 months ago

2 months ago

7 months ago

2 days ago

2 months ago

11 months ago

# Higher-Order Modal Logics (Lecture 5): Interacting with Modal Logics in the Coq Proof Assistant

Christoph Benz Müller and Bruno Woltzenlogel Paleo



# The Coq Proof Assistant

## Winner of the ACM Software System Award in 2013


CoqIDE

File Edit Navigation Try Tactics Templates Queries Display Compile Windows Help



\*scratch\* Modal.v ModalClassical.v GoedelGod-Scott.v

```

(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiomA1 : V (mforall p, (Positive (fun x: u => m~(p x))) m-> (m~ (Positive p))).
Axiom axiomA1b : V (mforall p, (m~ (Positive p)) m-> (Positive (fun x: u => m~ (p x))) ).

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiomA2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x) ) )).

(* Theorem T1: positive properties are possibly exemplified *)
Theorem theoremT1: V (mforall p, (Positive p) m-> dia (mexists x, p x) ).
Proof.
  intro.
  intro p.
  intro H1.
  proof_by_contradiction H2.
  apply not_dia_box_not_in H2.
  assert (H3: ((box (mforall x, m~ (p x))) w)). (* Lemma from Scott's notes *)
  box intro w1 R1.
  intro x.
  assert (H4: ((m~ (mexists x : u, p x)) w1)).
  box elim H2 w1 R1 G2.
  exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

  assert (H6: ((box (mforall x, (p x) m-> m~ (x m= x))) w)). (* Lemma from Scott's notes *)
  box intro w1 R1.
  intro x.
  intro H7.
  intro H8.
  box elim H3 w1 R1 G3.
  apply G3 with (x := x).

```

2 subgoals  
w : i  
p : u -> o  
H1 : Positive p w  
H2 : box (m~ (mexists x : u, p x)) w  
box (mforall x : u, m~ p x) w  
False

(1/2)  
(2/2)



# Typical Natural Deduction Rules

Interactive proving using basic tactics in Coq feels roughly like constructing a natural deduction proof

$$\begin{array}{ccc}
 \frac{A \vee B \quad \begin{array}{c} \overline{A} \\ \vdots \\ C \end{array} \quad \begin{array}{c} \overline{B} \\ \vdots \\ C \end{array}}{C} \vee_E & \frac{A \quad B}{A \wedge B} \wedge_I & \frac{\begin{array}{c} \overline{A} \\ \vdots \\ B \end{array} \quad h}{A \rightarrow B} \rightarrow_I^h \\
 \\
 \frac{A}{A \vee B} \vee_{I_1} & \frac{A \wedge B}{A} \wedge_{E_1} & \frac{B}{A \rightarrow B} \rightarrow_I \\
 \\
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 \\
 \frac{A[\alpha]}{\forall x.A[x]} \forall_I & \frac{\forall x.A[x]}{A[t]} \forall_E & \frac{A[t]}{\exists x.A[x]} \exists_I & \frac{\exists x.A[x]}{A[\beta]} \exists_E \\
 \\
 \neg A \equiv A \rightarrow \perp & \frac{\neg\neg A}{A} \neg\neg_E
 \end{array}$$

## Correspondence between Coq's tactics and ND's inference rules

Interactive proving using basic tactics in Coq feels roughly like constructing a natural deduction proof

| Rule                 | Tactic                                                     |
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| discharge assumption | <code>exact H. assumption.</code>                          |
| $\rightarrow_e$      | <code>cut. apply H.</code>                                 |
| $\rightarrow_i$      | <code>intro H.</code>                                      |
| $\forall_e$          | <code>apply H.</code>                                      |
| $\forall_i$          | <code>intro H.</code>                                      |
| $\neg_e$             | <code>apply H.</code>                                      |
| $\neg_i$             | <code>intro H.</code>                                      |
| <i>efq</i>           | <code>exfalso.</code>                                      |
| $\wedge_e$           | <code>destruct H as [H1 H2].</code>                        |
| $\wedge_i$           | <code>split.</code>                                        |
| $\vee_e$             | <code>destruct H as [H1   H2].</code>                      |
| $\vee_i$             | <code>left. right. classical_left. classical_right.</code> |
| $\exists_e$          | <code>destruct H as [a H1].</code>                         |
| $\exists_i$          | <code>exists t.</code>                                     |

**Automatic Tactics:** `auto`, `tauto`, `intuition`, `firstorder`, ...

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# The Proof/Type System behind Coq

- ▶ **Calculus of Inductive Constructions (CIC)**
  - ▶ Related to CoC (Calculus of Constructions) and  $\lambda C$
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## Curry-Howard Correspondence (1934, 1958; 1969, 1980)

Inference rules for minimal logic:

$$\frac{}{\Gamma, A \vdash A} \quad \frac{\Gamma \vdash B \quad \Gamma \vdash B \rightarrow C}{\Gamma \vdash C} \rightarrow_e \quad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \rightarrow C} \rightarrow_i^n$$

Typing rules for simply typed lambda calculus:

$$\frac{}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : (B \rightarrow C) \quad \Gamma \vdash N : B}{\Gamma \vdash (MN) : C} \quad \frac{\Gamma, x : B \vdash M : C}{\Gamma \vdash (\lambda x : B. M) : (B \rightarrow C)}$$

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Formulas  $\sim$  Types

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Normalization  $\sim$  Beta-Reduction

Is  $T$  provable?  $\sim$  Is  $T$  inhabited?

## Brouwer-Heyting-Kolmogorov “Interpretation” (1908, 1924; 1934; 1932)

### Constructivism - Intuitionism

A proof of  $B \wedge C$  is given by presenting a proof of  $B$  and a proof of  $C$ .

A proof of  $B \vee C$  is given by presenting either a proof of  $B$  or a proof of  $C$ .

A proof of  $B \rightarrow C$  is a construction that transforms  
any proof of  $B$  into a proof of  $C$ .

$\vdots$

## Two Completely Different Uses of Simply-Typed Lambda Calculus

- ▶ Languages used in previous lectures:
  - ▶ lambda **terms** provide syntax for higher-order **formulas**.
  - ▶ proof calculi have nothing to do with lambda calculus.
- ▶ Language of Minimal Logic seen in this lecture's previous slides:
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How do we go from minimal logic to higher-order logics?

We must generalize our types

## Polymorphism

terms may depend on types (System F or  $\lambda_2$ )

$$\overline{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma \vdash N : B \quad \Gamma \vdash M : B \rightarrow C}{\Gamma \vdash (MN) : C} \rightarrow_e \qquad \frac{\Gamma, x : B \vdash m : C}{\Gamma \vdash (\lambda x : B. m) : B \rightarrow C} \rightarrow_i^n$$



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$$\frac{\Gamma \vdash M : \forall X. A}{\Gamma \vdash (M T) : A[X \setminus T]} \forall_e \quad \frac{\Gamma \vdash M : A}{\Gamma \vdash (\Lambda X. M) : \forall X. A} \forall_i$$

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Simplification: replace the 4 rules above by the 2 more general rules below

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$$B \rightarrow C \equiv \forall \_ : B. C$$

But now we need a rule for judgements of the form:  $\Gamma \vdash (\forall x : B. C) : s$

$$\frac{\Gamma \vdash B : s \quad \Gamma, x : B \vdash C : \mathbf{Type}}{\Gamma \vdash (\forall x : B. C) : \mathbf{Type}} Prod$$

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## Dependent Types

types may depend on terms  $(\lambda P)$

$\vdots$

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$$\frac{\Gamma \vdash N : B \quad \Gamma \vdash M : (\forall x : B. C)}{\Gamma \vdash (M \ N) : C[x \setminus N]} \textit{App}$$

$$\frac{\Gamma \vdash A : \mathbf{Type} \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash (\forall x : A. B) : s} \textit{Prod}$$

## Pure Type Systems

A Pure Type System is a triple  $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ , where:

$\mathcal{S}$  is a set of sorts;  $\mathcal{A} \subseteq \mathcal{S} \times \mathcal{S}$  (axioms);  $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$  (*Prod* rules).

$$(Ax) \quad \frac{}{\vdash s_1 : s_2} \quad (s_1, s_2) \in \mathcal{A}$$

$$(Var) \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad x \notin \text{dom}(\Gamma)$$

$$(Prod) \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\forall x : A. B) : s_3} \quad (s_1, s_2, s_3) \in \mathcal{R}$$

$$(Abs) \quad \frac{\Gamma, x : A \vdash B : C \quad \Gamma \vdash (\forall x : A. C) : s}{\Gamma \vdash (\lambda x : A. B) : \forall x : A. C}$$

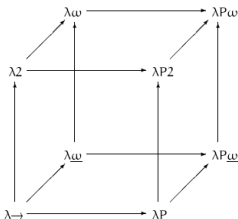
$$(App) \quad \frac{\Gamma \vdash A : (\forall x : B. C) \quad \Gamma \vdash D : B}{\Gamma \vdash (A D) : C[x := D]}$$

$$(Weak) \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}$$

$$(Conv) \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad B =_{\beta} B'$$



## Barendregt's Cube



The various type systems in the cube have  $\mathcal{A} = \{ (\mathbf{Type}, \mathbf{Kind}) \}$  and differ from each other w.r.t. the sorts allowed in the Prod rule:

$$(\text{Prod}) \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\forall x : A. B) : s_3} \quad (s_1, s_2, s_3) \in \mathcal{R}$$

- ▶  $\lambda \rightarrow$ :  $\mathcal{R} = \{ (\mathbf{Type}, \mathbf{Type}, \mathbf{Type}) \}$
- ▶  $\lambda P$ :  $\mathcal{R} = \{ (\mathbf{Type}, \mathbf{Type}, \mathbf{Type}), (\mathbf{Type}, \mathbf{Kind}, \mathbf{Kind}) \}$
- ▶  $\lambda 2$ :  $\mathcal{R} = \{ (\mathbf{Type}, \mathbf{Type}, \mathbf{Type}), (\mathbf{Kind}, \mathbf{Type}, \mathbf{Type}) \}$
- ▶ ...
- ▶  $\lambda C$ :  $\mathcal{R} =$   
 $\{ (\mathbf{Type}, \mathbf{Type}, \mathbf{Type}), (\mathbf{Kind}, \mathbf{Type}, \mathbf{Type}), (\mathbf{Type}, \mathbf{Kind}, \mathbf{Kind}), (\mathbf{Kind}, \mathbf{Kind}, \mathbf{Kind}) \}$

## Dialectica Interpretation

Not to be confused with the Curry-Howard Correspondence

Given a constructive proof  $\pi$  of a  $\Pi_2$  statement  $\forall x : T_1. \exists y : T_2. P(x, y)$ ,  
its dialectica interpretation  $D(\pi)$  is a function  $f : T_1 \rightarrow T_2$   
such that  $\forall x : T_1. P(x, f(x))$  is valid.

- ▶ Curry-Howard Correspondence: Proof of  $T$  is a program of type  $T$
- ▶ Dialectica: a program is extracted from a proof of  $\forall x. \exists y. P(x, y)$ .

Pure Coq can be considered intuitionistic/constructive, but . . .

There are different flavours of intuitionism/constructivism out there

Example:  $(\forall x : A. P(x)) \rightarrow (\exists x : A. P(x))$

Provable in Gentzen's Intuitionistic Calculi but not in Coq!

Coq does not assume that  $A$  is non-empty/inhabited

Coq's logic is a free logic

"Coq + Classical Library" is still a free logic

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- ▶ Sometimes ATPs are just not powerful/expressive enough.  
ITPs are less powerful; but are very expressive,  
and designed to receive instructions from the user.
- ▶ ATPs are complex and highly optimized, and hence bug-prone.  
ITPs rely on short and simple trusted code.  
(Currently, lots of research go into the integration of ATPs and ITPs.)
- ▶ Even when ATPs are powerful enough, their proofs can be ...
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# Issues with Fully Automated Reasoning

Proofs are hard to read and do not necessarily correspond to the informal proofs being checked

```

DemoMaterial — bash — 166x52

@SV8>@SV3)=false) | (((pe(^[SX0:mu,SX1:$i]: $false))@SV3)=true))),inference(prim_subst,[status(thm)],166:[bind(SV11,$thf(^[SV23:mu,SV24:$i]: $false))]])),
thf(84,plain,(![SV22:(mu=>($i>$o)),SV3:$i,SV8:(mu=>($i>$o))]: (((@SV8)((sk2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1)))@SV8))@((sk1_SY31@(^[SX0:mu,SX1:$i]: (~
((SV22@SX0)@SX1)))@SV8)@SV3)=true) | (((pe(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1)))@SV3)=true))),inference(prim_subst,[status(thm)],166
:[bind(SV11,$thf(^[SV20:mu,SV21:$i]: (~ ((SV22@SV20)@SV21)))])),
thf(85,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)=false) | (((pe@SV9)@SV4) = ((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)
@SY28)))@SV4)=false))),inference(fac_restr,[status(thm)],1561)),
thf(86,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)=true) | (((pe@SV9)@SV4) = ((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@
SY30)))@SV4)=false))),inference(fac_restr,[status(thm)],571)),
thf(87,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((~ ((pe@SV9)@SV4) | ((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)) | (~ ((~ ((pe@SV9)@SV4)) | (~ ((pe(^[SY27:mu,
SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)))=false) | (((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)=false))),inference(extcnf_equal_neg,[status(thm)],185))),
thf(89,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((~ ((pe@SV9)@SV4) | ((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)) | (~ ((~ ((pe@SV9)@SV4)) | (~ ((pe(^[SY29:mu,
SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)))=false) | (((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)=true))),inference(extcnf_equal_neg,[status(thm)],186))),
thf(92,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((~ ((pe@SV9)@SV4) | (~ ((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)))=false) | (((pe(^[SY27:mu,SY28:$i]: (~
((SV9@SY27)@SY28)))@SV4)=false))),inference(extcnf_or_neg,[status(thm)],187))),
thf(93,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((~ ((pe@SV9)@SV4) | ((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)))=false) | (((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@
SY29)@SY30)))@SV4)=true))),inference(extcnf_or_neg,[status(thm)],189))),
thf(96,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((~ ((pe@SV9)@SV4)) | (~ ((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)))=true) | (((pe(^[SY27:mu,SY28:$i]: (~ ((S
V9@SY27)@SY28)))@SV4)=false))),inference(extcnf_not_neg,[status(thm)],192))),
thf(97,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe@SV9)@SV4) | ((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)=true) | (((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)
@SY30)))@SV4)=true))),inference(extcnf_not_neg,[status(thm)],193))),
thf(100,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((~ ((pe@SV9)@SV4))=true) | ((~ ((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4))=true) | (((pe(^[SY27:mu,SY28:$i]: (~
((SV9@SY27)@SY28)))@SV4)=false))),inference(extcnf_or_pos,[status(thm)],196))),
thf(101,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe@SV9)@SV4)=true) | (((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)=true) | (((pe(^[SY29:mu,SY30:$i]: (~ ((SV
9@SY29)@SY30)))@SV4)=false))),inference(extcnf_or_pos,[status(thm)],197))),
thf(103,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe@SV9)@SV4)=false) | ((~ ((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4))=true) | (((pe(^[SY27:mu,SY28:$i]: (~
((SV9@SY27)@SY28)))@SV4)=false))),inference(extcnf_not_pos,[status(thm)],180))),
thf(105,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)=false) | (((pe@SV9)@SV4)=false) | (((pe(^[SY27:mu,SY28:$i]: (~ ((
SV9@SY27)@SY28)))@SV4)=false))),inference(extcnf_not_pos,[status(thm)],183))),
thf(107,plain,(![SV8:(mu=>($i>$o)),SV3:$i,SV22:(mu=>($i>$o))]: (((SV22@((sk2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1)))@SV8))@((sk1_SY31@(^[SX0:mu,SX1:$i]: (~
((SV22@SX0)@SX1)))@SV8)@SV3)=true) | (((pe@SV8)@SV3)=false) | ((pe(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1)))@SV3)=true))),inference(extcnf_not_neg,[status(thm)
],178))),
thf(108,plain,(![SV11:(mu=>($i>$o)),SV3:$i,SV15:(mu=>($i>$o))]: (((SV15@((sk2_SY33@SV3)@SV11)@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1)))@((sk1_SY31@SV11)@(^[SX0:mu,
SX1:$i]: (~ ((SV15@SX0)@SX1)))@SV3))=false) | (((pe(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1)))@SV3)=false) | (((pe@SV11)@SV3)=true))),inference(extcnf_not_pos,[statu
s(thm)],181))),
thf(109,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28)))@SV4)=false) | (((pe@SV9)@SV4)=false))),inference(sim,[status(thm)],110
5))),
thf(110,plain,(![SV4:$i,SV9:(mu=>($i>$o))]: (((pe@SV9)@SV4)=true) | (((pe(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30)))@SV4)=true))),inference(sim,[status(thm)],1101
))),
thf(111,plain,(![SV3:$i,SV8:(mu=>($i>$o))]: (((pe@SV8)@SV3)=false) | ((pe(^[SX0:mu,SX1:$i]: $true))@SV3)=true))),inference(sim,[status(thm)],176))),
thf(112,plain,(![SV11:(mu=>($i>$o)),SV3:$i]: (((pe(^[SX0:mu,SX1:$i]: $false))@SV3)=false) | ((pe@SV11)@SV3)=true))),inference(sim,[status(thm)],180))),
thf(113,plain,(((false)=true)),inference(fo_atp_e,[status(thm)],125,112,111,110,109,108,107,84,83,82,75,74,73,72,71,70,69,68,67,66,65,62,57,56,51,42,29))),
thf(114,plain,(false),inference(solved_all_splits,[status(thm)],1133)),
% SZS output end CNFRefutation

***** End of derivation protocol *****
***** no. of clauses in derivation: 97 *****
***** clause counter: 113 *****

% SZS status Unsatisfiable for ConsistencyWithoutFirstConjunctinD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibEQ:true,rAndEQ:true,use_choice:true,use_extuni:true,use_
extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,transla
tion:fof_full)
ontoleo:DemoMaterial cbenzmuellers []
```

## Embedding HOML in HOL

### Example

HOML formula

HOML formula in HOL

expansion,  $\beta\eta$ -conversion

expansion,  $\beta\eta$ -conversion

expansion,  $\beta\eta$ -conversion

$\Diamond \exists x G(x)$

valid  $(\Diamond \exists x G(x))_{\iota \rightarrow o}$

$\forall w_{\iota} (\Diamond \exists x G(x))_{\iota \rightarrow o} w$

$\forall w_{\iota} \exists u_{\iota} (rwu \wedge (\exists x G(x))_{\iota \rightarrow o} u)$

$\forall w_{\iota} \exists u_{\iota} (rwu \wedge \exists x Gxu)$

**Does this actually work in practice?**

Is it efficient ??

Is it user-friendly ??

- ▶ Challenges:
  - ▶ Can we hide the semantic embedding from the user?
  - ▶ Can we provide an interaction experience to the user that differs as little as possible from what he is already used to?
  - ▶ Can we reconstruct, step-by-step in Coq, precisely Scott's formulation of Gödel's argument?

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## Modal Logics in the Coq Proof Assistant

```
Parameter i: Type. (* Type for worlds *)
Parameter u: Type. (* Type for individuals *)
Definition o := i -> Prop. (* Type of modal propositions *)

Parameter r: i -> i -> Prop. (* Accessibility relation for worlds *)

Definition mnot (p: o)(w: i) := ~ (p w).
Notation "m~ p" := (mnot p) (at level 74, right associativity).

Definition mand (p q:o)(w: i) := (p w) /\ (q w).
Notation "p m /\ q" := (mand p q) (at level 79, right associativity).

Definition mor (p q:o)(w: i) := (p w) \/ (q w).
Notation "p m \/ q" := (mor p q) (at level 79, right associativity).

Definition mimplies (p q:o)(w:i) := (p w) -> (q w).
Notation "p m-> q" := (mimplies p q) (at level 99, right associativity).

Definition mequiv (p q:o)(w:i) := (p w) <-> (q w).
Notation "p m<-> q" := (mequiv p q) (at level 99, right associativity).
```

## Modal Logics in the Coq Proof Assistant

**Definition** A {t: Type}(p: t -> o)(w: i) := forall x, p x w.

**Notation** "'mforall' x , p" := (A (fun x => p))

(at level 200, x ident, right associativity) : type\_scope.

**Notation** "'mforall' x : t , p" := (A (fun x:t => p))

(at level 200, x ident, right associativity,

format "'[' 'mforall' '/' ' x : t , '/' ' p ']'")

: type\_scope.

**Definition** E {t: Type}(p: t -> o)(w: i) := exists x, p x w.

**Notation** "'mexists' x , p" := (E (fun x => p))

(at level 200, x ident, right associativity) : type\_scope.

**Notation** "'mexists' x : t , p" := (E (fun x:t => p))

(at level 200, x ident, right associativity,

format "'[' 'mexists' '/' ' x : t , '/' ' p ']'")

: type\_scope.

## Modal Logics in the Coq Proof Assistant

**Definition** box (p: o) := fun w => forall w1, (r w w1) -> (p w1).

**Definition** dia (p: o) := fun w => exists w1, (r w w1) /\ (p w1).

## Modal Logics in the Coq Proof Assistant

**Lemma** mp\_dia:

[mforall p, mforall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].

**Proof.** mv.

intros p q H1 H2. unfold dia. unfold dia in H1. unfold box in H2.

destruct H1 as [w0 [R1 H1]]. exists w0. split.

exact R1.

apply H2.

exact R1.

exact H1.

**Qed.**

## Modal Logics in the Coq Proof Assistant

```
Ltac box_i := let w := fresh "w" in let R := fresh "R"
              in (intro w at top; intro R at top).

Ltac box_elim H w1 H1 := match type of H with
  ((box ?p) ?w) => cut (p w1);
  [intros H1 | (apply (H w1); try assumption)] end.

Ltac box_e H H1:= match goal with | [ |- (_ ?w) ] => box_elim H w H1 end.

Ltac dia_e H := let w := fresh "w" in let R := fresh "R" in
  (destruct H as [w [R H]]); move w at top; move R at top).

Ltac dia_i w := (exists w; split; [assumption | idtac]).
```

## Modal Logics in the Coq Proof Assistant

**Lemma** mp\_dia:

[mforallall p, mforallall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].

**Proof.** mv.

intros p q H1 H2. unfold dia. unfold dia in H1. unfold box in H2.  
destruct H1 as [w0 [R1 H1]]. exists w0. split.

exact R1.

apply H2.

exact R1.

exact H1.

**Qed.**

**Lemma** mp\_dia:

[mforallall p, mforallall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].

**Proof.** mv.

intros p q H1 H2. dia\_e H1. dia\_i w0. box\_e H2 H3. apply H3. exact H1.

**Qed.**

# Natural Deduction Calculus

## Rules for Modalities

$$\frac{\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Box A} \Box_I$$

$$\frac{\Box A}{t : \boxed{\begin{array}{c} A \\ \vdots \end{array}}} \Box_E$$

$$\frac{t : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Diamond A} \Diamond_I$$

$$\frac{\Diamond A}{\beta : \boxed{\begin{array}{c} A \\ \vdots \end{array}}} \Diamond_E$$

### eigen-box condition:

$\Box_I$  and  $\Diamond_E$  are strong modal rules:

$\alpha$  and  $\beta$  must be fresh names for the boxes they access  
(in analogy to the eigen-variable condition for strong quantifier rules).  
Every box must be accessed by exactly one strong modal inference.

### boxed assumption condition:

assumptions should be discharged within  
the box where they are created.



## Modal Logics in the Coq Proof Assistant

**Lemma** mp\_dia:

[mforallall p, mforallall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].

**Proof.** mv.

intros p q H1 H2. dia\_e H1. dia\_i w0. box\_e H2 H3. apply H3. exact H1.

**Qed.**

$$\begin{array}{c}
 w \\
 \begin{array}{c}
 \begin{array}{c}
 \overline{\Diamond p} \quad 1 \quad \overline{\Box(p \rightarrow q)} \quad 2 \\
 \hline
 \Diamond_E \quad \Box_E
 \end{array} \\
 \\
 \begin{array}{c}
 w_0 \quad \begin{array}{c}
 \frac{p \quad p \rightarrow q}{q} \rightarrow_E
 \end{array}
 \end{array} \\
 \\
 \frac{\overline{\Diamond q} \quad \Diamond_I}{\Diamond p \rightarrow (\Box(p \rightarrow q)) \rightarrow (\Diamond q)} \rightarrow_I^1, \rightarrow_I^2 \\
 \hline
 \forall p. \forall q. \Diamond p \rightarrow (\Box(p \rightarrow q)) \rightarrow \Diamond q \quad \forall_I, \forall_I
 \end{array}
 \end{array}$$

# Modal Logics in the Coq Proof Assistant

## Part of Scott's Formulation of Gödel's Proof

```
(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: [ mforall p, (Positive p) m-> dia (mexists x, p x) ].
Proof. mv.
intro p. intro H1. proof_by_contradiction H2. apply not_dia_box_not in H2.
assert (H3: ((box (mforall x, m~ (p x))) w)). (* Scott *)
  box_i. intro x. assert (H4: ((m~ (mexists x : u, p x)) w0)).
    box_e H2 G2. exact G2.
  clear H2 R H1 w. intro H5. apply H4. exists x. exact H5.
assert (H6: ((box (mforall x, (p x) m-> m~ (x m= x))) w)). (* Scott *)
  box_i. intro x. intros H7 H8. box_elim H3 w0 G3. eapply G3. exact H7.
  assert (H9: ((Positive (fun x => m~ (x m= x))) w)). (* Scott *)
    apply (axiom2 w p (fun x => m~ (x m= x))). split.
      exact H1.
      exact H6.
  assert (H10: ((box (mforall x, (p x) m-> (x m= x))) w)). (* Scott *)
    box_i. intros x H11. reflexivity.
  assert (H11 : ((Positive (fun x => (x m= x))) w)). (* Scott *)
    apply (axiom2 w p (fun x => x m= x)). split.
      exact H1.
      exact H10.
  apply axiom1a in H9. contradiction.

Qed.
```

## T1 and C1

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# Natural Deduction Proofs for the Ontological Argument

## T2 (Partial)

$$\begin{array}{c}
 \frac{\psi(x)^6 \quad \frac{\psi(x) \rightarrow \Box P(\psi)}{\Box P(\psi)} \rightarrow_E}{\Box P(\psi)} \Pi_2 \\
 \frac{\Box P(\psi) \quad \frac{\Box P(\psi) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))}{\Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow_E}{\Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow_I^6 \\
 \frac{\Box \forall x.(G(x) \rightarrow \psi(x))}{\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow_I^6
 \end{array}$$

The proof is structured as follows:

- $\psi(x)^6$  is the starting assumption.
- $\Pi_2$  is a subproof showing  $\psi(x) \rightarrow \Box P(\psi)$ .
- $\Box P(\psi)$  is derived from  $\Pi_2$  and  $\psi(x)^6$  using  $\rightarrow_E$ .
- $\Pi_3$  is a subproof showing  $P(\psi) \rightarrow \forall x.(G(x) \rightarrow \psi(x))$ .
- $\Box P(\psi)^7$  is derived from  $\Pi_3$  and  $P(\psi)$  using  $\Box E$ .
- $\Box \forall x.(G(x) \rightarrow \psi(x))$  is derived from  $\Box P(\psi)$  and  $\Pi_3$  using  $\rightarrow_E$ .
- $\Box \forall x.(G(x) \rightarrow \psi(x))$  is derived from  $\Box P(\psi)$  and  $\Box \forall x.(G(x) \rightarrow \psi(x))$  using  $\rightarrow_I^7$ .
- $\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))$  is derived from  $\Box \forall x.(G(x) \rightarrow \psi(x))$  using  $\rightarrow_I^6$ .

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