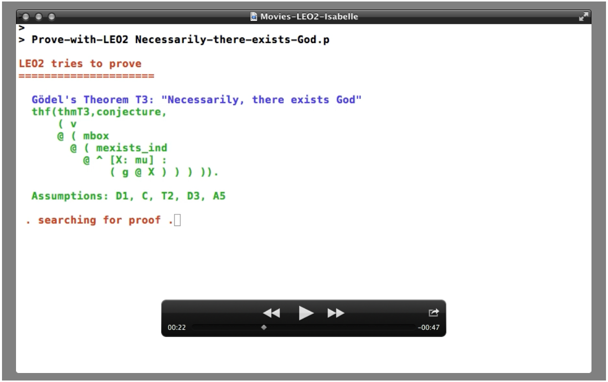
The Inconsistency in Gödel's Ontological Argument:

An Application of Mathematical Proof Assistants in Metaphysics

Christoph Benzmüller[1], FU Berlin & Stanford (CSLI/Cordula Hall)
jww: B. Woltzenlogel Paleo (& L. Paulson , C. Brown, G. Sutcliffe and many others!)

Mathematical Logic Seminar, Stanford University, May 17, 2016



Url to movie: `http://www.christoph-benzmueller.de/papers/Movies-LEO2-Isabelle.mov`

[1] Supported by DFG Heisenberg Fellowship BE 2501/9-1/2

Vision of Leibniz (1646–1716): *Calculemus!*

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo ... dicere: calculemus. (Leibniz, 1684)

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other ...: Let us calculate.

(Translation by Russell)

Required:
characteristica universalis and **calculus ratiocinator**

Talk Outline

A: HOL as a Universal (Meta-)Logic via Semantic Embeddings

B: New Knowledge on the Ontological Argument from HOL ATPs

C: Reconstruction of the Inconsistency of Gödel's Axioms

D: Recent Technical Improvements

E: Other Recent Work: Free Logic & Application in Category Theory

(F: Other Recent Work: Zalta'a Theory of Abstract Object)

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

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English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

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HOME / SCIENCE NEWS

Researchers say they used MacBook to prove Goedel's God theorem

God exists, say Apple fanboy scientists

With the help of just one MacBook, two Germans formalize a theorem that confirms the existence of God.

See more serious and funny news links at
`https://github.com/FormalTheology/GoedelGod/blob/master/Press/LinksToNews.md`

Part A:
HOL as a Universal (Meta-)Logic via Semantic Embeddings

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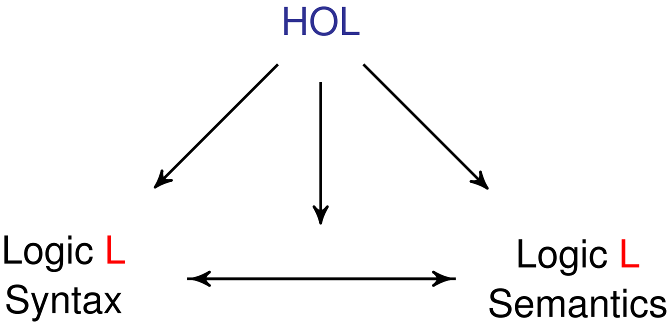
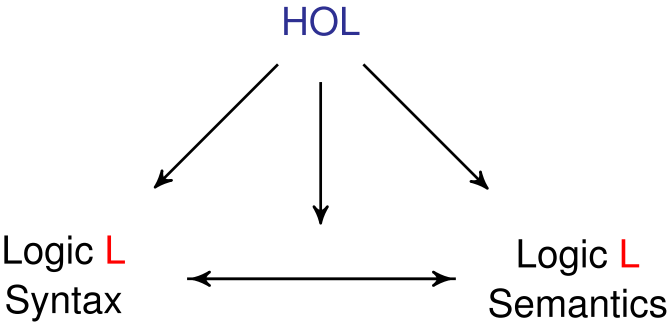


Examples for L we have already studied:
Modal Logics, Conditional Logics, Intuitionistic Logics, Access Control Logics, Nominal Logics, Multivalued Logics (SIXTEEN), Logics based on Neighborhood Semantics, (Mathematical) Fuzzy Logics, Paraconsistent Logics, Free Logic . . .

Works also for (first-order & higher-order) quantifiers



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Works also for (first-order & higher-order) quantifiers

Embedding Approach — Idea

HOL (meta-logic) φ ::=

Your-logic (object-logic) ψ ::=

Embedding of in

=

=

=

=

Embedding of meta-logical notions on in

valid =

satisfiable =

... =

Pass this set of equations to a higher-order automated theorem prover

Classical Higher-Order Logic (HOL)

Simple Types $\alpha ::= o \mid \iota \mid \mu \mid \alpha_1 \rightarrow \alpha_2$

HOL $s, t ::= c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid$
$(\neg_{o\rightarrow o}\, s_o)_o \mid (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (\forall_{(\alpha\rightarrow o)\rightarrow o}(\lambda x_\alpha t_o))_o$

(note: binder notation $\forall x_\alpha t_o$ as syntactic sugar for $\forall_{(\alpha\rightarrow o)\rightarrow o}(\lambda x_\alpha t_o)$)

HOL with Henkin semantics is (meanwhile) well understood

Origin [Church,JSymbLog,1940]

Henkin semantics [Henkin,JSymb.Log,1950]

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Extens./Intens. [BenzmüllerEtAl,JSymbLog,2004]

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Sound and complete provers do exists

interactive: Isabelle/HOL, PVS, HOL4, Hol Light, Coq/HOL, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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Embedding HOML in HOL

HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

f

- Kripke style semantics (possible world semantics)

$M, g, s \models \neg\varphi$ iff not $M, g, s \models \varphi$

$M, g, s \models \varphi \wedge \psi$ iff $M, g, s \models \varphi$ and $M, g, s \models \psi$

$\ldots$

$M, g, s \models \Box\varphi$ iff $M, g, u \models \varphi$ for all s with $r(s, u)$

$\ldots$

$M, g, s \models \forall x_\gamma\, \varphi$ iff $M, [d/x]g, s \models \varphi$ for all $d \in D_\gamma$

$\ldots$

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

[Muskens, HandbookOfModalLogic, 2006]

Embedding Approach — HOML in HOL (remember my talk at SRI in 2010!)

HOL $\quad s, t \quad ::= \quad c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\to o}$
(explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\to & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\, hdw \\
\exists & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, hdw \\
\Box & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \forall u_\mu\, (\neg r w u \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\mu\to o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

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$\rightarrow$	$=$	$\lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\neg\varphi w \vee \psi w)$
$\forall$	$=$	$\lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\, hdw$
$\exists$	$=$	$\lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, hdw$
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Embedding HOML in HOL

Example

HOML formula	$\Diamond\exists x G(x)$
HOML formula in HOL	$\text{valid}\,(\Diamond\exists x G(x))_{\mu\to o}$
expansion	$(\lambda\varphi\forall w_\mu \varphi\, w)(\Diamond\exists x G(x))_{\mu\to o}$
$\beta\eta$-normalisation	$\forall w_\mu((\Diamond\exists x G(x))_{\mu\to o}\, w)$
expansion	$\forall w_\mu(((\lambda\varphi_{\mu\to o}\lambda w_\mu \exists u_\mu\,(rwu \wedge \varphi u))\exists x G(x))_{\mu\to o}\, w)$
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syntactic sugar	$\forall w_\mu \exists u_\mu(rwu \wedge (\exists(\lambda x G(x)))_{\mu\to o} u)$
expansion	$\forall w_\mu \exists u_\mu(rwu \wedge ((\lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, hdw)(\lambda x G(x)))_{\mu\to o} u)$
$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu(rwu \wedge \exists x G x u)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu\to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu\to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Embedding HOML in HOL

Example

HOML formula	$\Diamond\exists x G(x)$
HOML formula in HOL	valid $(\Diamond\exists x G(x))_{\mu\to o}$
expansion	$(\lambda\varphi\forall w_\mu\varphi\, w)(\Diamond\exists x G(x))_{\mu\to o}$
$\beta\eta$-normalisation	$\forall w_\mu((\Diamond\exists x G(x))_{\mu\to o}\, w)$
expansion	$\forall w_\mu(((\lambda\varphi_{\mu\to o}\lambda w_\mu\exists u_\mu\,(\boldsymbol{r}wu \wedge \varphi u))\exists x G(x))_{\mu\to o}\, w)$
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Advantages of the Embedding Approach

1. Pragmatics and convenience:
    - implementing new provers made simple (even for not yet automated logics)
2. Availability:
    - simply reuse and adapt our existing encodings (THF, Isabelle/HOL, Coq)
3. Flexibility:
    - rapid experimentation with logic variations and logic combinations
4. Relation to labelled deductive systems:
    - extra-logical labels vs. intra-logical labels (here)
5. Relation to standard translation:
    - extra-logical translation vs. extended intra-logical translation (here)
6. Meta-logical reasoning:
    - various examples already exist, e.g. verification of modal logic cube
7. Direct calculi and user intuition:
    - possible: tactics on top of embedding, hiding of embedding
8. Soundness and completeness:
    - already proven for many non-classical logics (wrt Henkin semantics)
9. Cut-elimination:
    - generic indirect result, since HOL enjoys cut-elimination (Henkin semantics)

Advantage: 1. Pragmatics and convenience

implementing new provers made simple (even for not yet automated logics)

A very “Lean” Prover for HOML K

```
%----The base type $i (already built-in) stands here for worlds and
%----mu for individuals; $o (also built-in) is the type of Booleans
thf(mu_type,type,(mu:$tType)).
%----Reserved constant r for accessibility relation
thf(r,type,(r:$i>$i>$o)).
%----Modal logic operators not, or, and, implies, box, diamond
thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
thf(mnot,definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))).
thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
thf(mor,definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)|(Psi@W))))).
thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
thf(mand,definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
thf(mimplies,definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
thf(mbox,definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V)|(A@V))))).
thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
thf(mdia,definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V)&(A@V))))).
%----Quantifiers (constant domains) for individuals and propositions
thf(mforall_ind_type,type,(mforall_ind:(mu>$i>$o)>$i>$o)).
thf(mforall_ind,definition,(mforall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
thf(mforall_indset_type,type,(mforall_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mforall_indset,definition,(mforall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(A@X@W)))).
thf(mexists_ind_type,type,(mexists_ind:(mu>$i>$o)>$i>$o)).
thf(mexists_ind,definition,(mexists_ind = (^[A:mu>$i>$o,W:$i]:?[X:mu]:(A@X@W)))).
thf(mexists_indset_type,type,(mexists_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mexists_indset,definition,(mexists_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:?[X:mu>$i>$o]:(A@X@W)))).
%----Definition of validity (grounding of lifted modal formulas)
thf(v_type,type,(v:($i>$o)>$o)).
thf(mvalid,definition,(v = (^[A:$i>$o]:![W:$i]:(A@W)))).
```

TPTP THF0 syntax: [SutcliffeBenzmüller, J.Formalized Reasoning, 2010]

Advantage: 1. Pragmatics and convenience

implementing new provers made simple (even for not yet automated logics)

Approach is competitive

- First-order modal logic: see experiments in

  [BenzmüllerOttenRaths, ECAI, 2012]
  [BenzmüllerRaths, LPAR, 2013]
  [Benzmüller, ARQNL, 2014]

- Higher-order modal logics:

  There are no other systems yet!

Advantage: 2. Availability

simply reuse and adapt our existing encodings (THF, Isabelle/HOL, Coq)

HOML in Isabelle/HOL

QML.thy

```
abbreviation mnot   :: "σ⇒σ"  ("¬_"[52]53)
  where "¬φ ≡ λw. ¬φ(w)"
abbreviation mand   :: "σ⇒σ⇒σ"  (infixr"∧"51)
  where "φ∧ψ ≡ λw. φ(w)∧ψ(w)"
abbreviation mor    :: "σ⇒σ⇒σ"  (infixr"∨"50)
  where "φ∨ψ ≡ λw. φ(w)∨ψ(w)"
abbreviation mimp   :: "σ⇒σ⇒σ"  (infixr"→"49)
  where "φ→ψ ≡ λw. φ(w)⟶ψ(w)"
abbreviation mequ   :: "σ⇒σ⇒σ"  (infixr"↔"48)
  where "φ↔ψ ≡ λw. φ(w)⟷ψ(w)"
abbreviation mall   :: "('a⇒σ)⇒σ"  ("∀")
  where "∀Φ ≡ λw.∀x. Φ(x)(w)"
abbreviation mallB  :: "('a⇒σ)⇒σ"  (binder"∀"[8]9)
  where "∀x. φ(x) ≡ ∀φ"
abbreviation mexi   :: "('a⇒σ)⇒σ"  ("∃")
  where "∃Φ ≡ λw.∃x. Φ(x)(w)"
abbreviation mexiB  :: "('a⇒σ)⇒σ"  (binder"∃"[8]9)
  where "∃x. φ(x) ≡ ∃φ"
abbreviation meq    :: "μ⇒μ⇒σ"  (infixr"="52) -- "Equality"
  where "x=y ≡ λw. x = y"
abbreviation meqL    :: "μ⇒μ⇒σ"  (infixr"=ᴸ"52) -- "Leibniz Equality"
  where "x=ᴸy ≡ ∀φ. φ(x)→φ(y)"
abbreviation mbox   :: "σ⇒σ"  ("□_"[52]53)
  where "□φ ≡ λw.∀v. w r v ⟶ φ(v)"
abbreviation mdia   :: "σ⇒σ"  ("◇_"[52]53)
  where "◇φ ≡ λw.∃v. w r v ∧ φ(v)"
```

Documentation Sidekick Theories

Output Query Sledgehammer Symbols

See formalisations at `https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations`

Advantage: 3. Flexibility

rapid experimentation with logic variations and logic combinations

Postulating modal axioms or semantical constraints

HOL

'Syntactical' Sahlqvist axioms			'Semantical' constraints	
M:	valid $\forall\varphi(\Box^r\varphi \to \varphi)$	$\leftrightarrow$	$\forall x(\boldsymbol{r}xx)$	(reflexivity)
B:	valid $\forall\varphi(\varphi \to \Box^r\Diamond^r\varphi)$	$\leftrightarrow$	$\forall x\forall y(\boldsymbol{r}xy \to \boldsymbol{r}yx)$	(symmetry)
D:	valid $\forall\varphi(\Box^r\varphi \to \Diamond^r\varphi)$	$\leftrightarrow$	$\forall x\exists y(\boldsymbol{r}xy)$	(serial)
4:	valid $\forall\varphi(\Box^r\varphi \to \Box^r\Box^r\varphi)$	$\leftrightarrow$	$\forall x\forall y\forall z(\boldsymbol{r}xy \wedge \boldsymbol{r}yz \to \boldsymbol{r}xz)$	(transitivity)
5:	valid $\forall\varphi(\Diamond^r\varphi \to \Box^r\Diamond^r\varphi)$	$\leftrightarrow$	$\forall x\forall y\forall z(\boldsymbol{r}xy \wedge \boldsymbol{r}xz \to \boldsymbol{r}yz)$	(euclidean)

Advantage: 3. Flexibility

rapid experimentation with logic variations and logic combinations

Possibilist vs. Actualist Quantification

$\forall \; = \lambda h_{\gamma\to(\mu\to o)} \lambda w_\mu \forall d_\gamma \, hdw$ (constant domains)

becomes

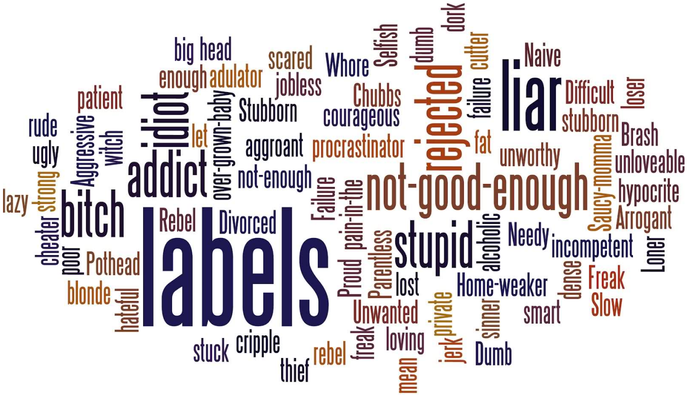
$\forall^{va} = \lambda h_{\gamma\to(\mu\to o)} \lambda w_\mu \forall d_\gamma \, (\textbf{ExInW} \, dw \to hdw)$ (varying domains)

where **ExInW** is an existence predicate

(additional axioms: non-empty domains, denotation of constants & functions)

Advantage: 4. Relation to labelled deductive systems

extra-logical labels vs. intra-logical labels (here)



$$\Diamond\exists x G(x)^{\text{worldlabel}} \quad \longrightarrow \quad ((\Diamond\exists x G(x))_{\mu\to o}\ \text{worldlabel}_{\mu})$$

Advantage: 5. Relation to standard translation

extra-logical translation vs. extended intra-logical translation (here)

[BenzmüllerPaulson, LogicaUniversalis, 2013]
[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

Intra-logical realisation of the standard translation

$$
\begin{array}{ll}
 & (\Box\phi)\ a \\
\longrightarrow & ((\Box\phi)_{\mu\to o}\ a) \\
\longrightarrow & (((\lambda\varphi_{\mu\to o}\lambda w_\mu \forall u_\mu\ (\neg \boldsymbol{r} wu \vee \varphi u))\ \phi)_{\mu\to o}\ a) \\
\longrightarrow & (\forall u_\mu\ (\neg rau \vee \phi_{\mu\to o}\ u)
\end{array}
$$

We have extended this also for first-order and higher-order quantifiers!

$$
\begin{array}{ll}
 & (\forall x\, \phi(x))\ a \\
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\longrightarrow & ((\forall(\lambda x\, \phi(x)))_{\mu\to o}\ a) \\
\longrightarrow & (((\lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\ hdw)(\lambda x\, \phi(x)))_{\mu\to o}\ a) \\
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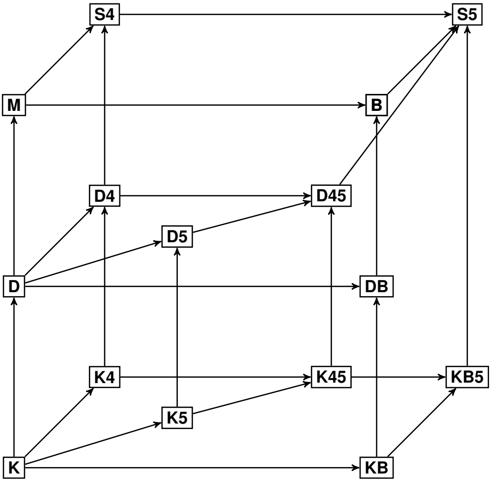
\end{array}
$$

Advantage: 6. Meta-logical reasoning

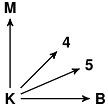
various examples already exist, e.g. verification of modal logic cube

[Benzmüller, FestschriftWalther, 2010]
[BenzmüllerClausSultana, PxTP, 2015]



S5 $\equiv$ **M5** $\equiv$ **MB5** $\equiv$ **M4B5** $\equiv$ **M45** $\equiv$ **M4B** $\equiv$ **D4B** $\equiv$ **D4B5** $\equiv$ **DB5**

M: $\Box P \to P$
B: $P \to \Box\Diamond P$
D: $\Box P \to \Diamond P$
4: $\Box P \to \Box\Box P$
5: $\Diamond P \to \Box\Diamond P$



KB5 $\equiv$ **K4B5** $\equiv$ **K4B**

Verification of cube in less than 1 minute in Isabelle/HOL

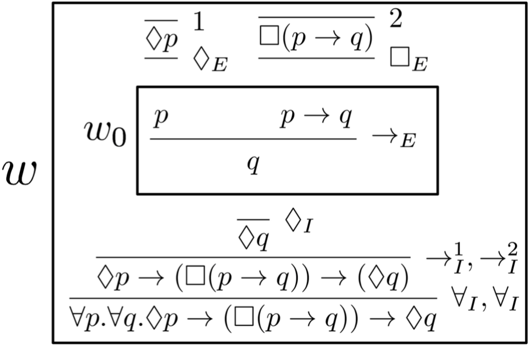
Advantage: 7. Direct calculi and user intuition

abstract level tactics (here in Coq) on top of embedding, hiding of embedding

[BenzmüllerWoltzenlogelPaleo, CSR'2015]

```
Lemma mp_dia:
  [mforall p, mforall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].
```

```
Proof. mv.
intros p q H1 H2. dia_e H1. dia_i w0. box_e H2 H3. apply H3. exact H1.
Qed.
```



Advantage: 8. Soundness and completeness

already proven for many non-classical logics (wrt Henkin semantics)

Soundness and Completeness

$\models^{L} \varphi \quad \text{iff} \quad Ax \models^{HOL}_{\text{Henkin}} \mathit{valid}\, \varphi_{\mu\to o}$

Logic L:

- Higher-order Modal Logics [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]
- First-order Multimodal Logics [BenzmüllerPaulson, LogicaUniversalis, 2013]
- Propositional Multimodal Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Quantified Conditional Logics [Benzmüller, IJCAI, 2013]
- Propositional Conditional Logics [BenzmüllerEtAl., AMAI, 2012]
- Intuitionistic Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- Access Control Logics [Benzmüller, IFIP SEC, 2009]
- Logic Combinations [Benzmüller, AMAI, 2011]
- ... more is on the way ... including:
  - Description Logics
  - Nominal Logics
  - Multivalued Logics (SIXTEEN)
  - Logics based on Neighborhood Semantics
  - (Mathematical) Fuzzy Logics
  - Paraconsistent Logics

Advantage: 9. Cut-elimination

generic indirect result, since HOL enjoys cut-elimination (Henkin semantics)

Soundness and Completeness

$\models^{L} \varphi \quad \text{iff} \quad Ax \models^{HOL}_{\text{Henkin}} \mathit{valid}\, \varphi_{\mu \to o}$

Logic L:

- Higher-order Modal Logics [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]
- First-order Multimodal Logics [BenzmüllerPaulson, LogicaUniversalis, 2013]
- Propositional Multimodal Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
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Soundness and Completeness and Cut-elimination

$$\models^{L} \varphi \quad \text{iff} \quad Ax \models^{HOL}_{\text{Henkin}} \mathit{valid}\, \varphi_{\mu\to o} \quad \text{iff} \quad Ax \vdash^{\text{HOL}}_{\text{cut-free}} \mathit{valid}\, \varphi_{\mu\to o}$$

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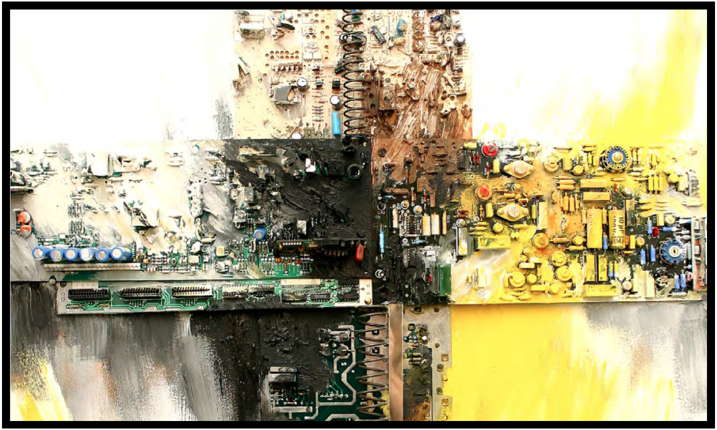

Part B:
New Knowledge on the Ontological Argument from HOL ATPs

Vision of Leibniz (1646–1716): *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
characteristica universalis and **calculus ratiocinator**

Our Contribution: Towards Computational Metaphysics

Ontological argument for the existence of God

- Focus on Gödel's modern version in higher-order modal logic
- Experiments with HO provers and embedding approach

Different interests in ontological arguments

- Philosophical: Boundaries of metaphysics & epistemology
- Theistic: Successful argument could convince atheists?
- Ours: Computational metaphysics (Leibniz' vision)

Related work: only for Anselm's simpler argument

- first-order ATP PROVER9 [OppenheimerZalta, 2011]
- interactive proof assistant PVS [Rushby, 2013]

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A Long History

pros and cons

… Anselm v. C. Gaunilo … Th. Aquinas …… Descartes Spinoza Leibniz … Hume Kant … Hegel … Frege … Hartshorne Malcolm Lewis Plantinga Gödel …

Anselm's notion of God (Proslogion, 1078):

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical, deductive reasoning:

"God exists."

$$\exists x G(x)$$

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To show by logical, deductive reasoning:

"Necessarily, God exists."

$\Box\exists x G(x)$

Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ • At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess. } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset MN(\exists y) G(y)$

" $\supset N(\exists y) G(y)$ M = possibility

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x) G(x)$ means the system of all pos. prop. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl

~~hence~~ { $x = x$ is positive
~~and~~ { $x \neq x$ is negative

But if a system S of pos. prop. were incons. It would mean, that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It may also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ neg. contrary At. or the existence of pos. prop.

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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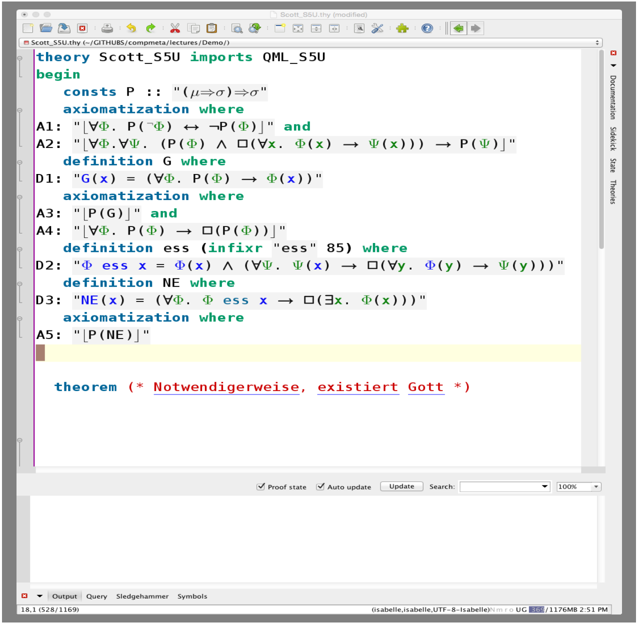
second-order quantifiers

Gödel's God in TPTP THF

```
>
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p

Leo-II tries to prove
=====================

  Goedel's Theorem T3: "Necessarily, God exists"
  thf(thmT3,conjecture,
      ( v
      @ ( mbox
        @ ( mexists_ind
          @ ^ [X: mu] :
              ( g @ X ) ) ) )).

  Assumptions: D1, C, T2, D3, A5

 . searching for proof ..

 ********************
 *   Proof found    *
 ********************
 % SZS status Theorem for Notwendigerweise-existiert-Gott.p

 . generating proof object
```

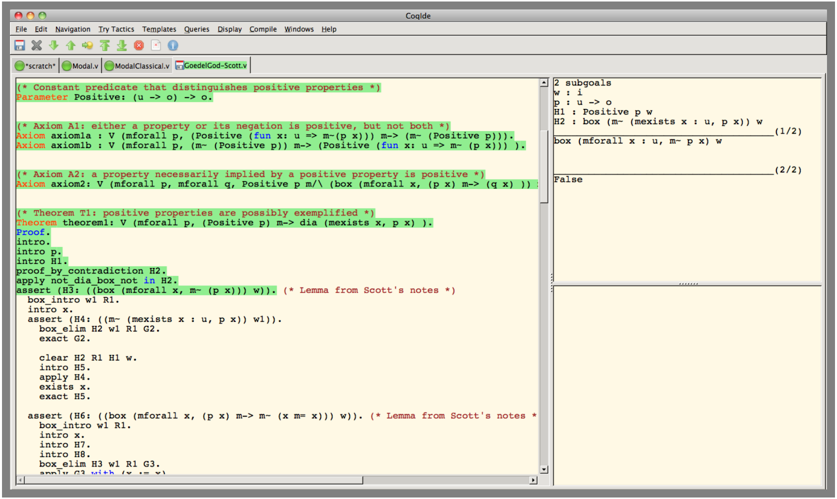
Url to movie: `http://www.christoph-benzmueller.de/papers/Movie1.mov`

Gödel's God in Isabelle/HOL



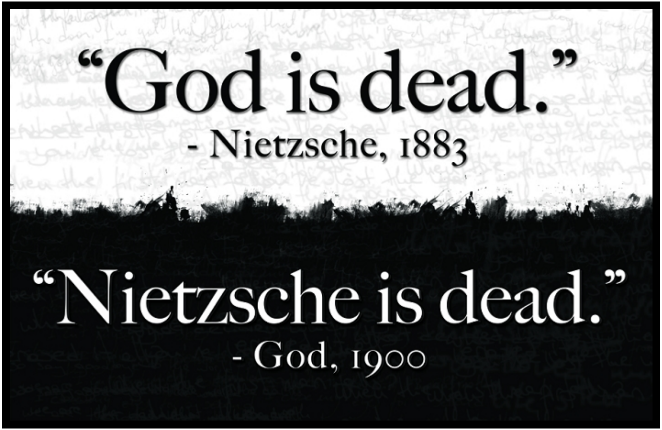
Url to movie: `http://www.christoph-benzmueller.de/papers/DemoMovieLehrpreis2.mov`

Gödel's God in Coq



See verifiable Coq document at:

`https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq`



Findings from our study

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathrm{ess}\, \phi X \dot{\supset} \dot{\Box}\dot{\exists}Y_\mu.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \dot{=} Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

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T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\,\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
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		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\,\dot\forall X_\mu.\,(g_{\mu\to\sigma}X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\,\dot\forall Y_\mu.\,(g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\,\lambda X_\mu.\,\dot\forall\psi_{\mu\to\sigma}.\,(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\,(\phi Y \dot\supset \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \,\dot{\equiv}\, \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\,\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \,\dot{\supset}\, \psi X)) \,\dot{\supset}\, p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Box}p\phi]$						
D2	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \,\dot{\supset}\, \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \,\dot{\supset}\, (ess_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$NE_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma} NE_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \,\dot{\supset}\, \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \,\dot{\supset}\, ($						
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \,\dot{\supset}\, (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **A1 and A2**
  - **A1($\supset$) and A2**
- **notion of quantification**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot\wedge\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu.\, (\phi Y \,\dot\supset\, \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \,\dot\supset\, (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

C: "Possibly, God exists" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **T1, D1, A3**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

T2: "Being God-like is an ess. of any God-like being" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **A1, D1, A4, D2**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.(e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

T3: "Necessarily, God exists"
proved by LEO-II and Satallax

- **in logic: KB**
- **from assumptions:**
  - **D1, C, T2, D3, A5**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

For logic K we got a countermodel by Nitpick

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \,\dot\wedge\, \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.(e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g \ldots X \dot\supset ($						
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

Summary

- proof verified and automated
- KB is sufficient (critisized logic S5 not needed!)
- possibilist and actualist quantifiers (individuals)
- exact dependencies determined experimentally
- ATPs have found alternative proofs
  e.g. self-identity $\lambda x(x = x)$ is not needed

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

Consistency check: Gödel vs. Scott

- Scott's assumptions are consistent; shown by Nitpick
- Gödel's assumptions are **inconsistent**; shown by LEO-II **(new philosophical result?)**

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall \phi_{\mu\to\sigma}. \dot\forall \psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu. (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu. \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall \phi_{\mu\to\sigma}. p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma} X]$						
A4	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot\forall \phi_{\mu\to\sigma}. (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma} \mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma} X]$						
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall \phi_{\mu\to\sigma}. \dot\forall X_\mu. (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu. \dot\forall Y_\mu. (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \dot\forall\psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to} \ldots \lambda\phi_{\mu\to\sigma}.\, \lambda$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

Further Results

- Monotheism holds
- God is flawless

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding						
A1	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\text{.}\, \dot\neg$						
A2	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, \dot\forall\psi_{\mu\to\sigma}\text{.}\, (p_{(\mu\to\sigma)\to}$						
T1	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists$						
D1	$g_{\mu\to\sigma} = \lambda X_\mu\text{.}\, \dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu\text{.}\, g_{\mu\to\sigma}X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p$						
D2	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\text{.}\, \lambda$						
T2	$[\dot\forall X_\mu\text{.}\, g_{\mu\to\sigma}X \,\dot\supset\, (\text{ess}_{(\mu\to\sigma)}$						
D3	$\text{NE}_{\mu\to\sigma} = \lambda X_\mu\text{.}\, \dot\forall\phi_{\mu\to\sigma}\text{.}\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\text{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu\text{.}\, g_{\mu\to\sigma}X]$						
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, \dot\forall X_\mu\text{.}\, (g_{\mu\to\sigma}X \,\dot\supset\, (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot\supset\, \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu\text{.}\, \dot\forall Y_\mu\text{.}\, (g_{\mu\to\sigma}X \,\dot\supset\, (g_{\mu\to\sigma}Y \,\dot\supset\, X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\text{.}\, \lambda X_\mu\text{.}\, \dot\forall\psi_{\mu\to\sigma}\text{.}\, (\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu\text{.}\, (\phi Y \,\dot\supset\, \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Modal Collapse (Sobel)

$$\forall\varphi(\varphi \supset \Box\varphi)$$

- **proved by LEO-II and Satallax**
- **for possibilist and actualist quantification (ind.)**

Main critique on Gödel's ontological proof:

- **there are no contingent truths**
- **everything is determined / no free will**

Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}. \dot\forall\psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu. (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu. \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. (\mathrm{ess}\, \phi X \dot\supset \dot\Box\dot\exists Y_\mu. \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma} \mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma} X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	17.9/—	3.3/3.2	—/—
		3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	16.5/—	0.0/0.0	—/—
		3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
			KB	THM	—/—	0.0/3.3	—/—
		3, A5	KB	THM	—/—	—/—	—/—
		3, A5	KB	SAT	—/—	—/—	7.3/7.4
			KB	UNS	7.5/7.8	—/—	—/—
		3, A5	KB	UNS	—/—	—/—	—/—

Observation

- good performance of ATPs
- excellent match between argumentation granularity in papers and the reasoning strength of the ATPs

Avoiding the Modal Collapse: Recent Variants

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's Ontological Proof Revisited *

C. Anthony Anderson and Michael Gettings

University of California, Santa Barbara
Department of Philosophy

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Magari and others on Gödel's ontological proof

Petr Hájek
Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic
e-mail: hajek@uivt.cas.cz

1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes

(betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,
sondern glauben, daß ein Gott sei.
(Kant, Nachlaß)

1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

A New Small Emendation of Gödel's Ontological Proof

Petr Hájek

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

Understanding Gödel's Ontological Argument

Frode Bjørdal

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

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Computer-supported Clarification of Controversy
1st World Congress on Logic and Religion, 2015

Results Obtained with Fully Automated Reasoners

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

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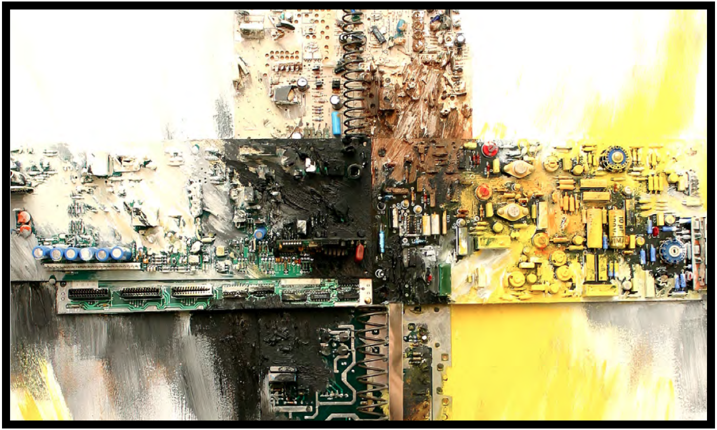
Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

Leibniz (1646–1716)

characteristica universalis *and* ***calculus ratiocinator***

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

But: **Intuitive proofs/models are needed to convince philosophers**



Part C:
Reconstruction of the Inconsistency of Gödel's Axioms

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \land \Box\forall x[\phi(x) \to \psi(x)]) \to P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \to \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \to \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \to \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \boxed{\cancel{\phi(x) \land}}\ \forall\psi(\psi(x) \to \Box\forall y(\phi(y) \to \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \to G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \to \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

Difference to Gödel (who omits this conjunct)

Inconsistency (Gödel): Proof by LEO-II in KB

DemoMaterial — bash — 166×52

```
@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$true))),inference(prim_subst,[status(thm)],[66:[bind(SV11,$thf(^[SV23:mu,SV24:$i]: $false))]])).
 thf(84,plain,(![SV22:(mu>($i>$o)),SV3:$i,SV8:(mu>($i>$o))]: ((((SV8@(((sK2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV8))@(((sK1_SY31@(^[SX0:mu,SX1:$i]: (
~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(prim_subst,[status(thm)],[66
:[bind(SV11,$thf(^[SV20:mu,SV21:$i]: (~ ((SV22@SV20)@SV21))))]])).
 thf(85,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | ((((p@SV9)@SV4) = ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)
@SY28))))@SV4))=$false))),inference(fac_restr,[status(thm)],[56])).
 thf(86,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true) | ((((p@SV9)@SV4) = ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@
SY30))))@SV4))=$false))),inference(fac_restr,[status(thm)],[57])).
 thf(87,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,
SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_equal_neg,[status(thm)],[85])).
 thf(89,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY29:mu,
SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_equal_neg,[status(thm)],[86])).
 thf(92,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (
~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_neg,[status(thm)],[87])).
 thf(93,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@
SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_neg,[status(thm)],[89])).
 thf(96,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))=$true) | (((p@(^[SY27:mu,SY28:$i]: (~ ((S
V9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_neg,[status(thm)],[92])).
 thf(97,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4))=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)
@SY30))))@SV4)=$true))),inference(extcnf_not_neg,[status(thm)],[93])).
 thf(100,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ ((p@SV9)@SV4))=$true) | ((~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))=$true) | (((p@(^[SY27:mu,SY28:$i]:
 (~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_pos,[status(thm)],[96])).
 thf(101,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV
9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_pos,[status(thm)],[97])).
 thf(103,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$false) | ((~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))=$true) | (((p@(^[SY27:mu,SY28:$i]: (~
 ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_pos,[status(thm)],[100])).
 thf(105,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4)=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((
SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_pos,[status(thm)],[103])).
 thf(107,plain,(![SV8:(mu>($i>$o)),SV3:$i,SV22:(mu>($i>$o))]: ((((SV22@(((sK2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV8))@(((sK1_SY31@(^[SX0:mu,SX1:$i]:
 (~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(extcnf_not_neg,[status(thm
)],[78])).
 thf(108,plain,(![SV11:(mu>($i>$o)),SV3:$i,SV15:(mu>($i>$o))]: ((((SV15@(((sK2_SY33@SV3)@SV11)@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1)))))@(((sK1_SY31@SV11)@(^[SX0:mu,
SX1:$i]: (~ ((SV15@SX0)@SX1))))@SV3))=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1))))@SV3)=$false) | (((p@SV11)@SV3)=$true))),inference(extcnf_not_pos,[statu
s(thm)],[81])).
 thf(109,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4)=$false))),inference(sim,[status(thm)],[10
5])).
 thf(110,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(sim,[status(thm)],[101]
)).
 thf(111,plain,(![SV3:$i,SV8:(mu>($i>$o))]: ((((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $true))@SV3)=$true))),inference(sim,[status(thm)],[76])).
 thf(112,plain,(![SV11:(mu>($i>$o)),SV3:$i]: ((((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$false) | (((p@SV11)@SV3)=$true))),inference(sim,[status(thm)],[80])).
 thf(113,plain,((($false)=$true)),inference(fo_atp_e,[status(thm)],[25,112,111,110,109,108,107,84,83,82,75,74,73,72,71,70,69,68,67,66,65,62,57,56,51,42,29])).
 thf(114,plain,($false),inference(solved_all_splits,[solved_all_splits(join,[])],[113])).
% SZS output end CNFRefutation

%***** End of derivation protocol ****
%***** no. of clauses in derivation: 97 ****
%***** clause counter: 113 ****

% SZS status Unsatisfiable for ConsistencyWithoutFirstConjunctinD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibEQ:true,rAndEQ:true,use_choice:true,use_extuni:true,use_
extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,transl
ation:fof_full)
ontoleo:DemoMaterial cbenzmueller$
```

Inconsistency (Gödel): Verification in Isabelle/HOL (KB)

GoedelGodWithoutConjunctInEss_KB.thy

GoedelGodWithoutConjunctInEss_KB.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
                   and A2:  "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"

-- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (∃ Φ))]"                    by (metis A1a A2)

  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"

-- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(∀(λx.( (λy.λw. False) ess x)))]"          by (metis ess_def)

  definition NE  where "NE x = ∀(λΦ. Φ ess x m→ □ (∃ Φ))"
  axiomatization where sym: "x r y ⟶ y r x"

-- {* Exemplification of necessary existence is not possible. *}
  lemma Lemma2: "[m¬ (◇ (∃ NE))]"                   by (metis sym Lemma1 NE_def)

  axiomatization where A5:  "[P NE]"

-- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
  lemma False                                           by (metis A5 T1 Lemma2)
end
```

Documentation Sidekick Theories

Output Query Sledgehammer Symbols

11,1 (477/1095) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 263/347MB 17:18

Inconsistency (Gödel): Verification in Isabelle/HOL (K)

GoedelGodWithoutConjunctInEss_K.thy

GoedelGodWithoutConjunctInEss_K.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"
  definition NE  where                   "NE x     = ∀(λΦ. Φ ess x m→ □ (∃ Φ))"
  axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
                   and A2:  "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"

-- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (∃ Φ))]"                          by (metis A1a A2)

-- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(∀(λx.((λy.λw. False) ess x)))]"                by (metis ess_def)

  axiomatization where A5:  "[P NE]"

-- {* Now the inconsistency follows from A5, Lemma1, NE_def and T1 *}
  lemma False
  -- {* sledgehammer [remote_leo2] *}
                                              by (metis A5 Lemma1 NE_def T1)
end
```

Documentation Sidekick Theories

Output Query Sledgehammer Symbols

21,7 (980/982) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 302/343MB 11:37

Inconsistency (Gödel): Informal Argument (in KB and K)

Def. D2* $\phi \; ess. \; x \leftrightarrow \cancel{\phi(x) \wedge} \; \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$

Inconsistency (Gödel): Informal Argument (in KB and K)

Def. D2* $\phi\ ess.\ x \leftrightarrow \cancel{\phi(x) \wedge}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset\ ess.\ x)$

Inconsistency (Gödel): Informal Argument (in KB and K)

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Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

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Axiom A5 $P(NE)$

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Axiom A5 $P(NE)$

- by T1, A5: $\Diamond\exists x[NE(x)]$

Inconsistency (Gödel): Informal Argument (in KB and K)

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Axiom A5 $P(NE)$

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Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Inconsistency (Gödel): Informal Argument (in KB and K)

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Axiom A5 $P(NE)$

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Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

- $\Diamond\exists x[\forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$

Inconsistency (Gödel): Informal Argument (in KB and K)

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- $\Diamond\exists x[\forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$
- $\Diamond\exists x[\emptyset\ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$

Inconsistency (Gödel): Informal Argument (in KB and K)

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Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

- $\Diamond\exists x[\forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$
- $\Diamond\exists x[\emptyset\ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$
- by L1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$

Inconsistency (Gödel): Informal Argument (in KB and K)

Def. D2* $\phi\ ess.\ x \leftrightarrow \cancel{\phi(x) \wedge}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset\ ess.\ x)$

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Axiom A5 $P(NE)$

- by T1, A5: $\Diamond\exists x[NE(x)]$

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

- $\Diamond\exists x[\forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$
- $\Diamond\exists x[\emptyset\ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$
- by L1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$
- by def. of $\emptyset$ $\Diamond\exists x[\top \rightarrow \Box\bot]$

Inconsistency (Gödel): Informal Argument (in KB and K)

Def. D2* $\phi\ ess.\ x \leftrightarrow \cancel{\phi(x) \wedge} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

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Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

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- by T1, A5: $\Diamond\exists x[NE(x)]$

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- by L1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$
- by def. of $\emptyset$ $\Diamond\exists x[\top \rightarrow \Box\bot]$
- $\Diamond\exists x[\Box\bot]$

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Def. D2* $\phi\ ess.\ x \leftrightarrow \cancel{\phi(x) \wedge}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

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- $\Diamond\exists x[\emptyset\ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$
- by L1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$
- by def. of $\emptyset$ $\Diamond\exists x[\top \rightarrow \Box\bot]$
- $\Diamond\exists x[\Box\bot]$
- $\Diamond\Box\bot$

Inconsistency (Gödel): Informal Argument (in KB and K)

Def. D2* $\phi \ ess.\ x \leftrightarrow \cancel{\phi(x) \wedge} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

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Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi \ ess.\ x \rightarrow \Box\exists y\phi(y)]$

- $\Diamond\exists x[\forall\phi[\phi \ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$
- $\Diamond\exists x[\emptyset \ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$
- by L1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$
- by def. of $\emptyset$ $\Diamond\exists x[\top \rightarrow \Box\bot]$
- $\Diamond\exists x[\Box\bot]$
- $\Diamond\Box\bot$

Inconsistency $\bot$

Gödel's Manuscript: Identifying the Inconsistent Axioms

Ontologischer Beweis — Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$• At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \, \text{Ess.}x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ — because it follows from the nature of the property

Th. $G(x) \supset G \, \text{Ess.}x$

Df. $E(x) \equiv (\varphi)[\varphi \, \text{Ess}\, x \supset N \exists x \, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x) G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x) G(x) \supset MN(\exists y)\, G(y)$

" $\supset N(\exists y)\, G(y)$ — M = possibility

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x) G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl[illegible]

~~[illegible]~~ $x = x$ is positive

~~[illegible]~~ $x \neq x$ is negative

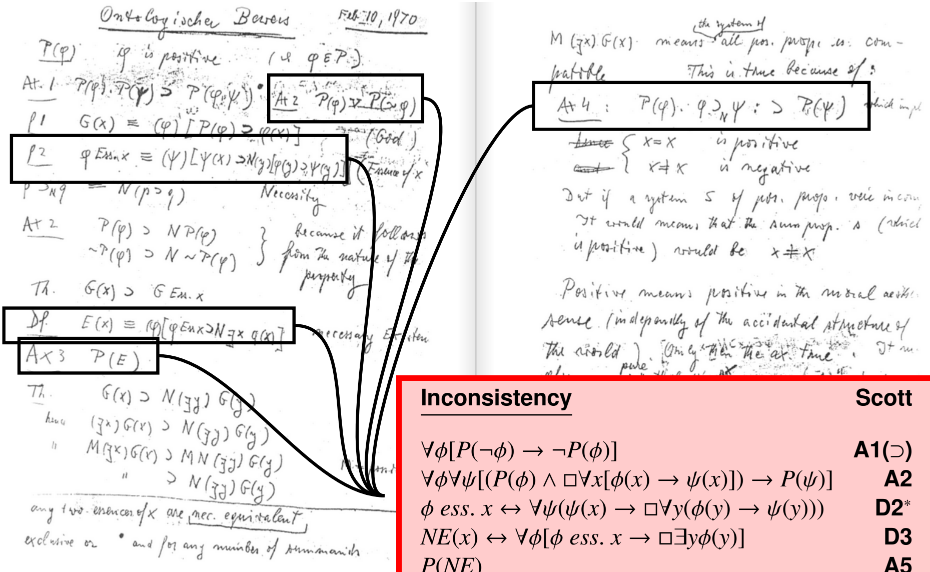
But if a system S of pos. props. were incon[illegible] it would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth[illegible] sense (independently of the accidental structure of the world). Only then the ax. true. It m[illegible] also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret. [illegible] proof

If φ pos. not: $(x) N \sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ neg. contrary Ax or the existence of pos. prop.

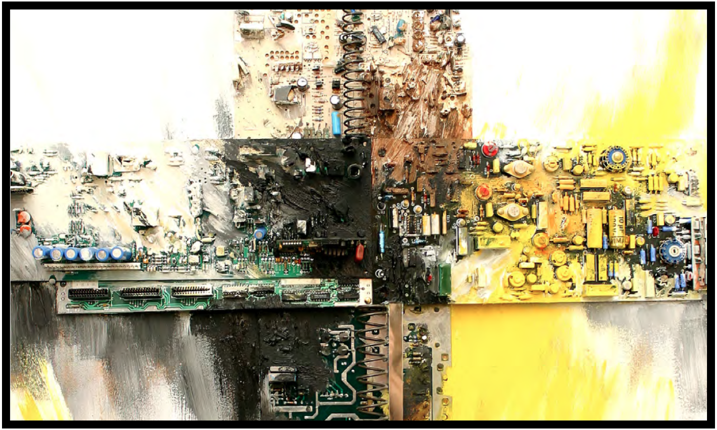
* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

Gödel’s Manuscript: Identifying the Inconsistent Axioms



Inconsistency **Scott**

$\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$	**A1(⊃)**
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$	**A2**
$\phi\ ess.\ x \leftrightarrow \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$	**D2***
$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$	**D3**
$P(NE)$	**A5**



Part D:
Recent Technical Improvements

Usability: More Intuitive Syntax for Embedded Logics in Isabelle

```
definition ess :: "(μ ⇒ σ) ⇒ μ ⇒ σ" (infixr "ess" 85) where
  "Φ ess x = Φ x m∧ ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"
```

```
definition ess (infixr "ess" 85) where
  "Φ ess x = Φ(x) ∧ (∀Ψ. Ψ(x) → □(∀y. Φ(y) → Ψ(y)))"
```

Improved Embedding of Modal Logic S5: S5U

Modal Logic S5

- Reflexivity: $\forall x.(r\ x\ x)$
- Symmetry: $\forall x.\forall y.(r\ x\ y) \to (r\ y\ x)$
- Transitivity: $\forall x.\forall y.\forall z.(r\ x\ y) \wedge (r\ y\ z) \to (r\ x\ z)$

Modal Logic S5U: with universal accessibility

- Universality: $\forall x.\forall y.(r\ x\ y)$

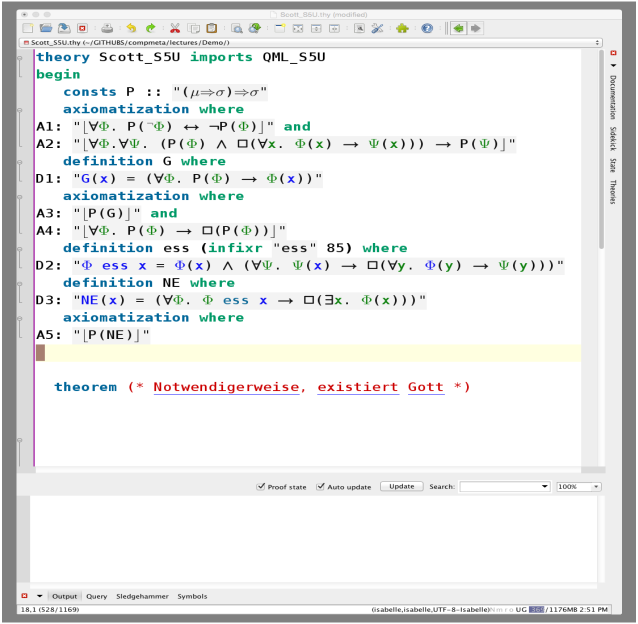
S5

$$\Box\varphi \equiv \lambda w.\forall v.\cancel{r(w,v) \to}\varphi(v) \qquad \text{and} \qquad \Diamond\varphi \equiv \lambda w.\exists v.\cancel{r(w,v) \wedge}\varphi(v)$$

S5U

$$\Box\varphi \equiv \lambda w.\forall v.\varphi(v) \qquad \text{and} \qquad \Diamond\varphi \equiv \lambda w.\exists v.\varphi(v)$$

Gödel's God in Isabelle/HOL



Url to movie: `http://www.christoph-benzmueller.de/papers/DemoMovieLehrpreis2cut.mov`

Part E:
Free Logic (Scott) & Application in Category Theory

Free Logic: Elegant Approach to Deal with Undefinedness

Principle 1: Bound individual variables range over domain $E \subset D$

Principle 2: The domain E may be empty.

Principle 3: The values of terms and free variables are in D, not necessarily in E only.

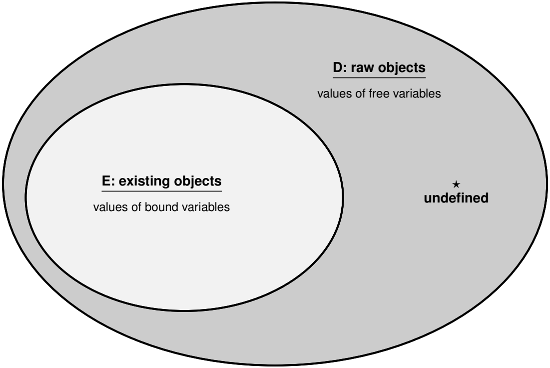
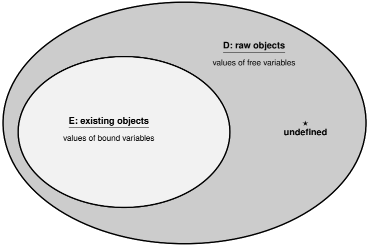
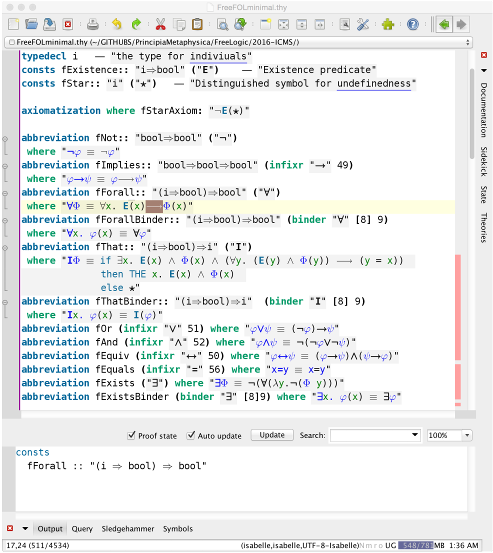


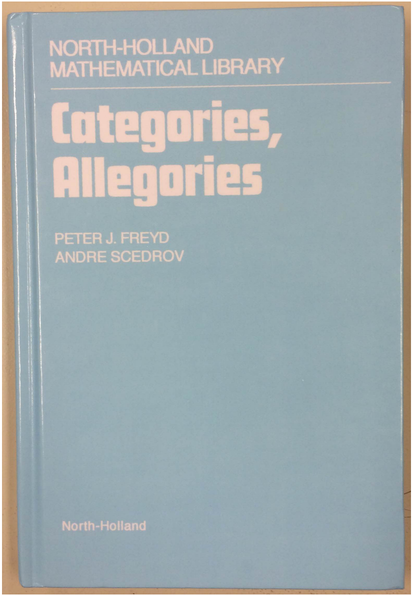
Figure: Illustration of the Semantical Domains of Free Logic

Free Logic in HOL: Application in Category Theory (jww Dana Scott)





Free Logic in HOL: Application in Category Theory (jww Dana Scott)



1.1. BASIC DEFINITIONS

The theory of CATEGORIES is given by two unary operations and a binary partial operation. In most contexts lower-case variables are used for the 'individuals' which are called *morphisms* or *maps*. The values of the operations are denoted and pronounced as:

$\Box x$ *the source of* x ,

$x\Box$ *the target of* x ,

xy *the composition of* x *and* y .

The axioms:

A1 xy *is defined* *iff* $x\Box = \Box y$,

A2a $(\Box x)\Box = \Box x$ *and* $\Box(x\Box) = x\Box$, A2b

A3a $(\Box x)x = x$ *and* $x(x\Box) = x$, A3b

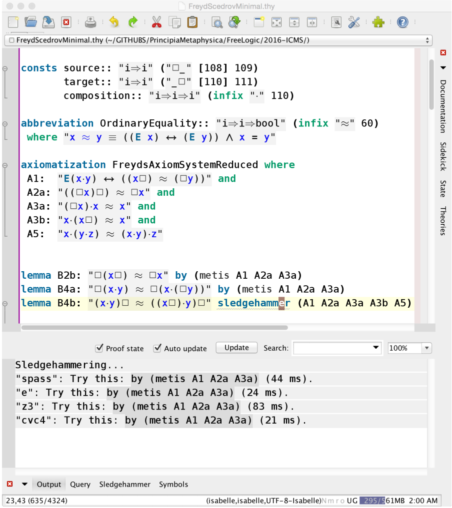
A4a $\Box(xy) = \Box(x(\Box y))$ *and* $(xy)\Box = ((x\Box)y)\Box$, A4b

A5 $x(yz) = (xy)z$.

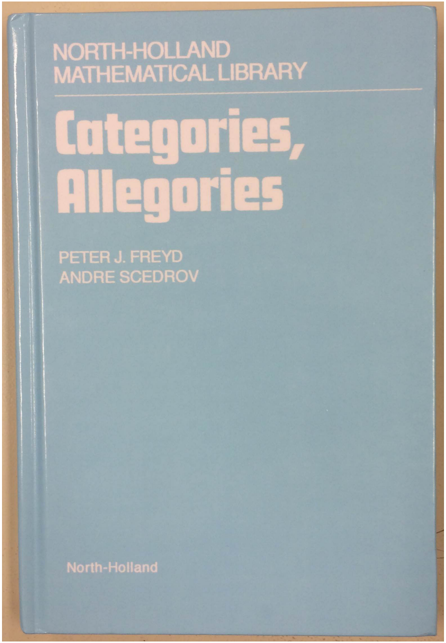
1.11. The ordinary equality sign = will be used only in the symmetric sense, to wit: if either side is defined then so is the other and they are equal. A theory, such as this, built on an ordered list of partial operations, the domain of definition of each given by equations in the previous, and with all other axioms equational, is called an ESSENTIALLY ALGEBRAIC THEORY.

1.12. We shall use a venturi-tube $\succeq$ for *directed equality* which means: if the left side is defined then so is the right and they are equal. The axiom that $\Box(xy) = \Box(x(\Box y))$ is equivalent, in the presence of the earlier axioms, with $\Box(xy) \succeq \Box x$ as can be seen below.

1.13. $\Box(\Box x) = \Box x$ because $\Box(\Box x) = \Box((\Box x)\Box) = (\Box x)\Box = \Box x$. Similarly $(x\Box)\Box = x\Box$.



Free Logic in HOL: Application in Category



1.1. BASIC DEFINITIONS

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~~A4~~ $\Box(xy) \cancel{=} \Box(x(\Box y))$ *and* $(xy)\Box \cancel{=} ((x\Box)y)\Box$, ~~A45~~

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Conclusion

Overall Achievements

- significant contribution towards a **Computational Metaphysics**
- **novel results** contributed by **HOL-ATPs**
- infrastructure can be adapted for **other logics and logic combinations**
- **basic technology works well**; however, improvements still needed

Relevance (wrt foundations and applications)

- Philosophy, AI, Computer Science, Computational Linguistics, Maths

Related work: only for Anselm's simpler argument

- first-order ATP PROVER9 [OppenheimerZalta, 2011]
- interactive proof assistant PVS [Rushby, 2013]

Ongoing/Future work

- (Awarded) Lecture course **Computational Metaphysics** at FU Berlin
- Landscape of verified/falsified ontological arguments
- You may contribute: `https://github.com/FormalTheology/GoedelGod.git`