Calculemus! — Progress in Universal Logic Reasoning and Computational Metaphysics

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Presentation Structure

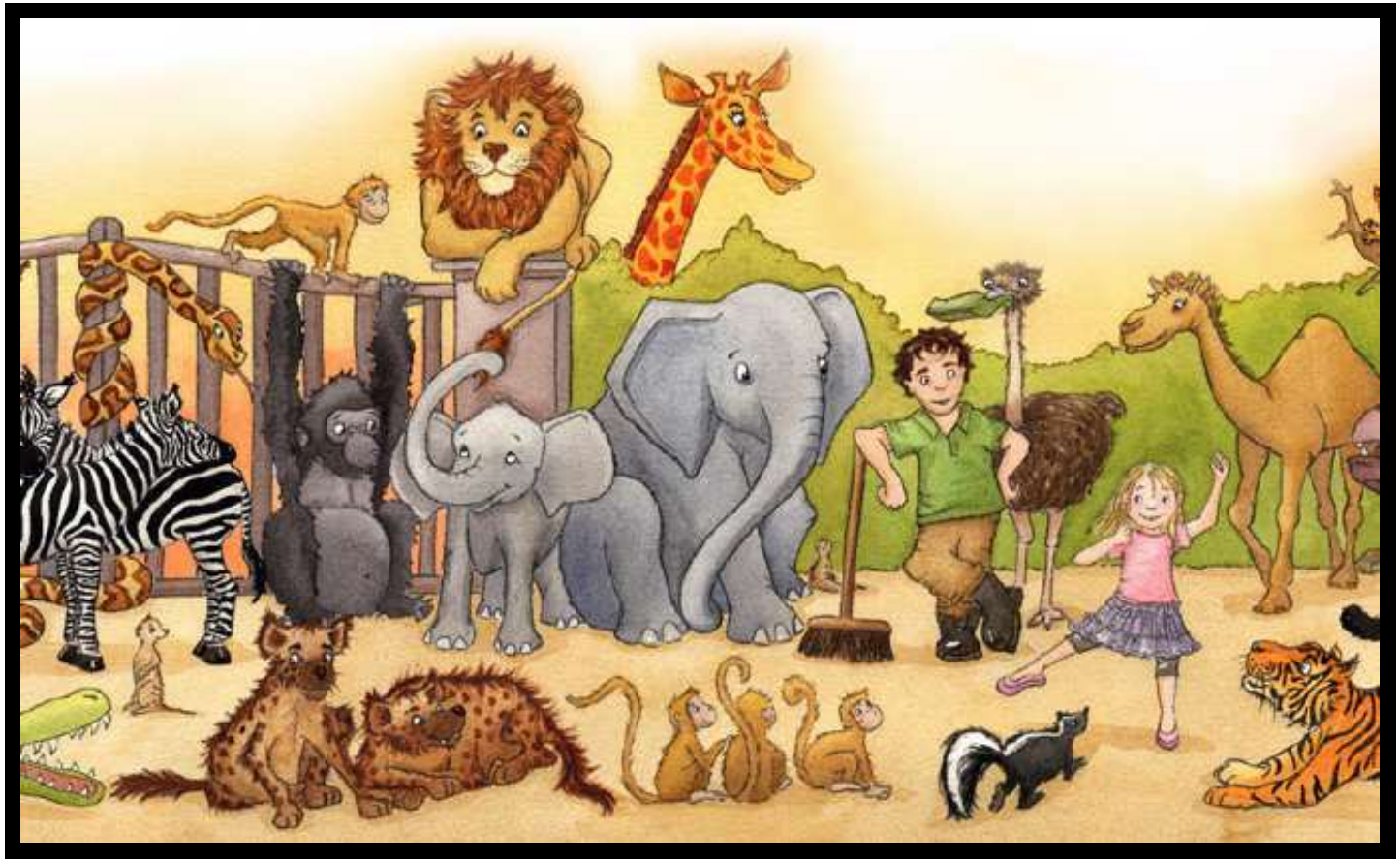
A Logic Zoo

B **Universal Logic Reasoning in HOL**

C **Computational Metaphysics**

D Conclusion

E Research and Teaching Profile



Part A
Logic Zoo

"If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis."

(Leibniz, 1677)

(A) Logic Zoo

Classical Logic, of order

0. Propositional Logic
1. First-order Logic
2. Second-order Logic

...

n. Higher-order Logic

Non-Classical Logics

- Intuitionistic/Constructive Logics (incl. Univalent Foundations)
- Modal Logics, Conditional Logics, Temporal Logis, Spatial Logics
- Many-valued Logics
- Paraconsistent Logics
- Free Logics, Inclusive Logics
- Logics for special applications: Ethics, Social Choice, Legal Reasoning, ...
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Example Application in Maths:

- Pythagorean Triples Problem solved by SAT-Solving in 2016

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Example Applications in Maths:

- First-order Set Theories: ZF, ZFC, ...

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Example Applications in Maths:

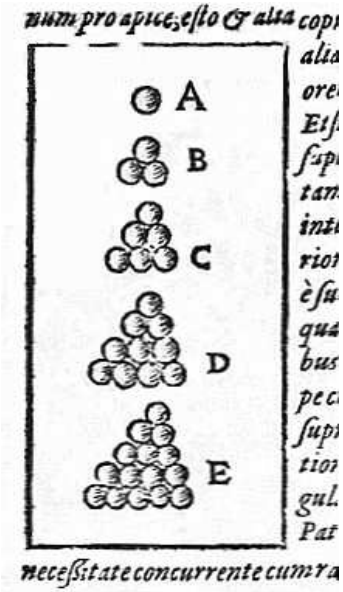
- Four-Colour Theorem: Formal verification by Gonthier in 2005 (Coq)
- Kepler's Conjecture: Formal verification by Hales in 2014 (HOL-light)

(A) Logic Zoo: Keplers Conjecture — Higher-order Logic —

Kepler (1571-1630)

The most compact way of stacking balls of the same size in space is a pyramid.

$$V\_ = \frac{\pi}{\sqrt{18}} \approx 74\%$$



- Proved in 1998 by Hales: 300 page proof, with code and data
- Submitted to the Annals of Mathematics: referees gave up to verify it all
- **Flyspeck project (completed in 2014):**
  - A formal verification of the proof in HOL Light
  - 27,223 proved theorems, 228 definitions, 30 person-years
- **Recent work of Cezary Kalyszik & Josef Urban:**
  Combination of **Machine Learning & Automated Theorem Proving**
  Result: more than 50% of the proofs can already be fully automated

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Own Research: Utilise Higher-order Logic (HOL) as Meta-Logic

This turns

- HOL into a (quite) universal logic
- HOL provers into (quite) universal reasoners

Applications in Maths, CS, AI, Philosophy, Computational Linguistics, ...

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Higher-order Logic: Don't be afraid of it!

Paradoxes (e.g. Russel's Paradox) eliminated by **Types**
Incompleteness avoided by **Henkin Semantics**

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Example Application in Metaphysics/Philosophy: Rest of this talk!

Necessarily, God exists: $\Box\exists x.Gx$

Kurt Gödel's definition of God: $G\,x \;:=\; \forall\Phi.Positive\ \Phi \rightarrow \Phi x$

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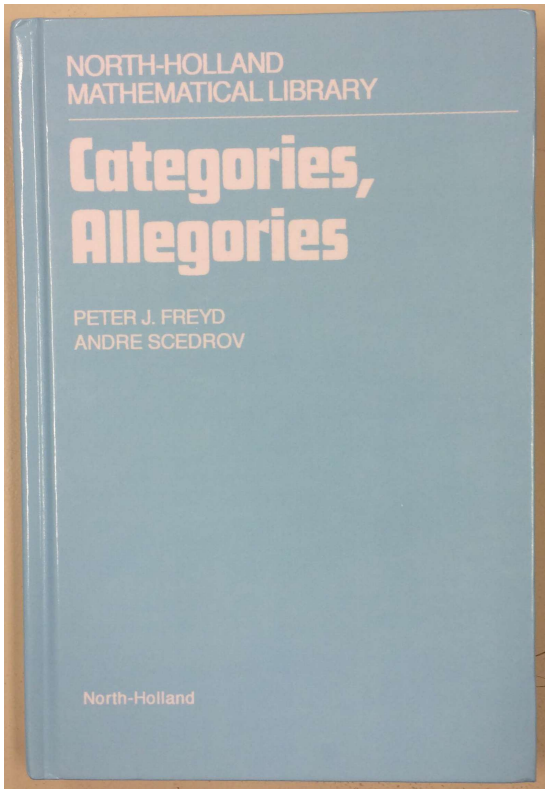
Example Application in Maths: (own jww with Dana Scott)

- Free Logics are well suited for for modeling **undefinedness and partiality**
- **Flaw detected in category theory textbook** by Freyd and Scedrov (1990)

(A) Logic Zoo: Theory Exploration in Category Theory — Free First-order Logic —



1.1. BASIC DEFINITIONS

The theory of CATEGORIES is given by two unary operations and a binary partial operation. In most contexts lower-case variables are used for the 'individuals' which are called *morphisms* or *maps*. The values of the operations are denoted and pronounced as:

$\square x$ *the source of* x ,

$x\square$ *the target of* x ,

xy *the composition of* x *and* y .

The axioms:

A1 xy *is defined iff* $x\square = \square y$,

A2a $(\square x)\square = \square x$ *and* $\square(x\square) = x\square$, A2b

A3a $(\square x)x = x$ *and* $x(x\square) = x$, A3b

A4a $\square(xy) = \square(x(\square y))$ *and* $(xy)\square = ((x\square)y)\square$, A4b

A5 $x(yz) = (xy)z$.

- Joint work with: Dana Scott (Berkeley) and students at FU Berlin
- Logics: Free Logic — well suited for modeling undefinedness and partiality
- **Results:**
  - development of 6 related (equivalent) axiom systems for category theory
  - from Monoids . . . via Scott's (1977) axiom system . . . to Freyd/Scedrov (1990)
  - **for Freyd and Scedrov (1990) we revealed some flaws/issues**
- Further Reading: [Benzmüller&Scott, ICMS'2016], [Benzmüller&Scott, arXiv'2016]

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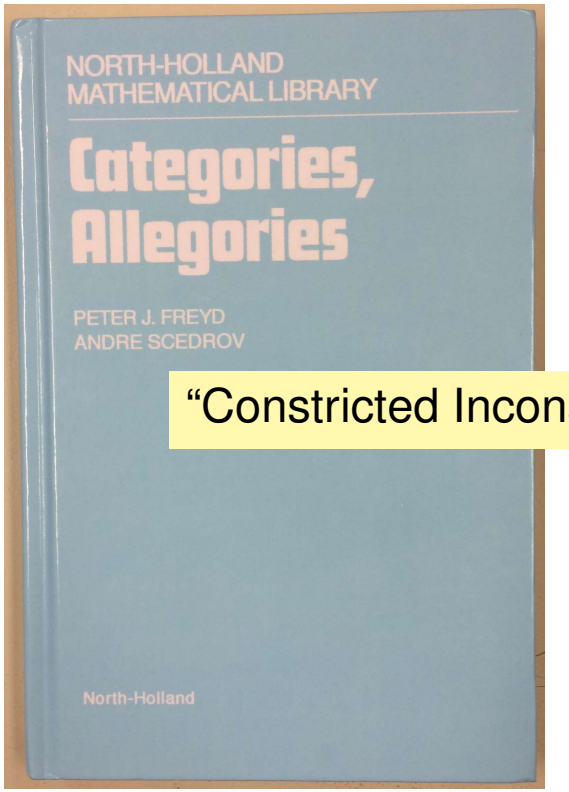

"Constricted Inconsistency" or "Missing Axioms/Conditions"

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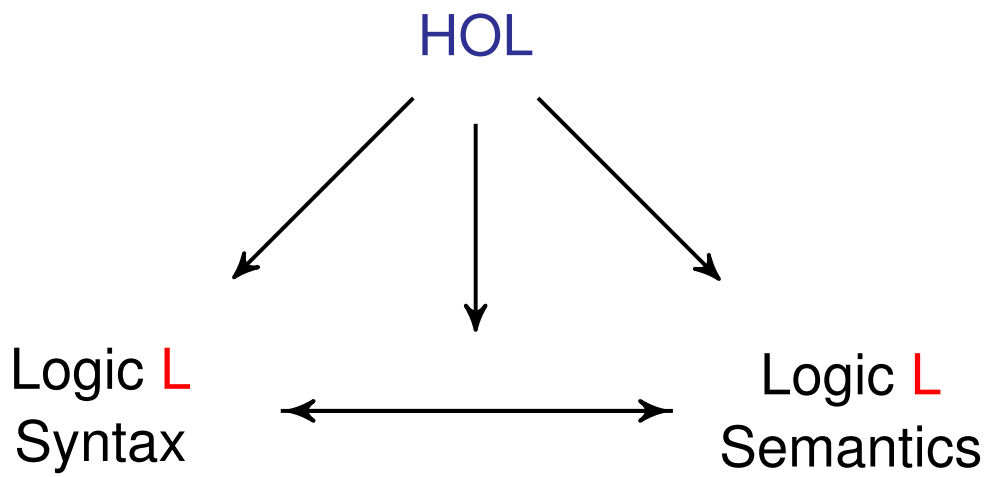
Part B
Universal Logic Reasoning in HOL via Shallow Semantical Embeddings

jww:
Larry Paulson (Cambridge, UK), Bruno Woltzenlogel-Paleo (ANU, Australia),
Alex Steen and Max Wisniewski (both FU Berlin)

(B) Universal Logic Reasoning in HOL



Examples for L we have already studied:
Modal Logics, Description Logics, Conditional Logics, Intuitionistic Logics, Access Control Logics, Nominal Logics, Multivalued Logics (SIXTEEN), Logics based on Neighborhood Semantics, (Mathematical) Fuzzy Logics, Paraconsistent Logics, Free Logic ...

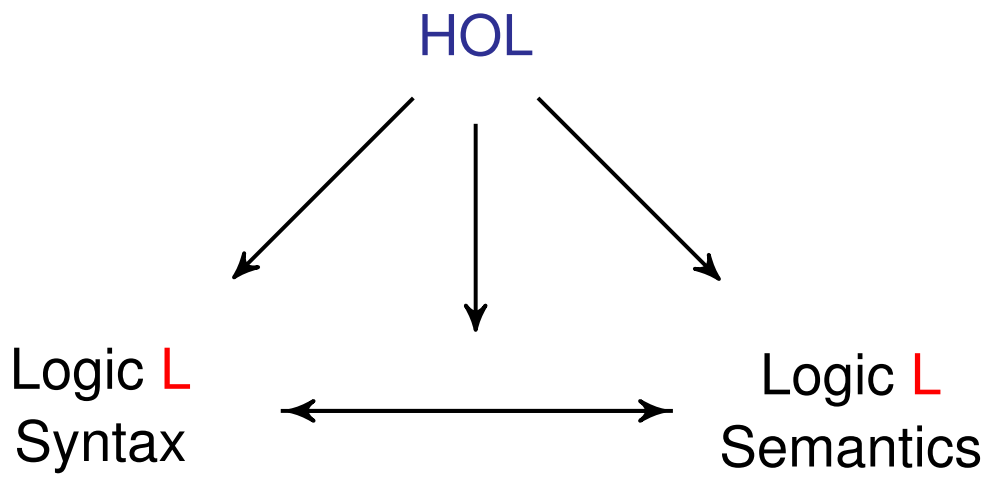
Embedding works also for quantifiers (first-order & higher-order)

HOL provers become universal logic reasoning engines!

interactive: Isabelle/HOL, PVS, HOL4, Hol Light, Coq/HOL, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

(B) Universal Logic Reasoning in HOL



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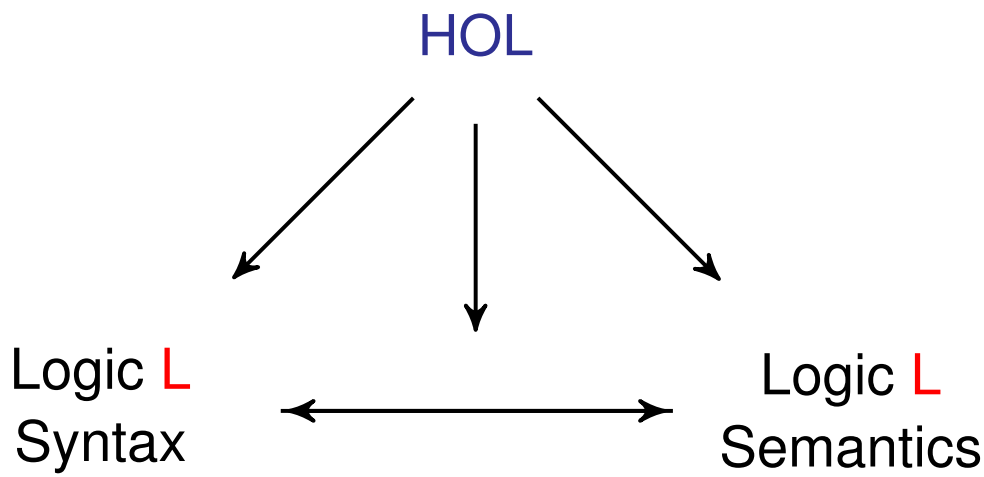

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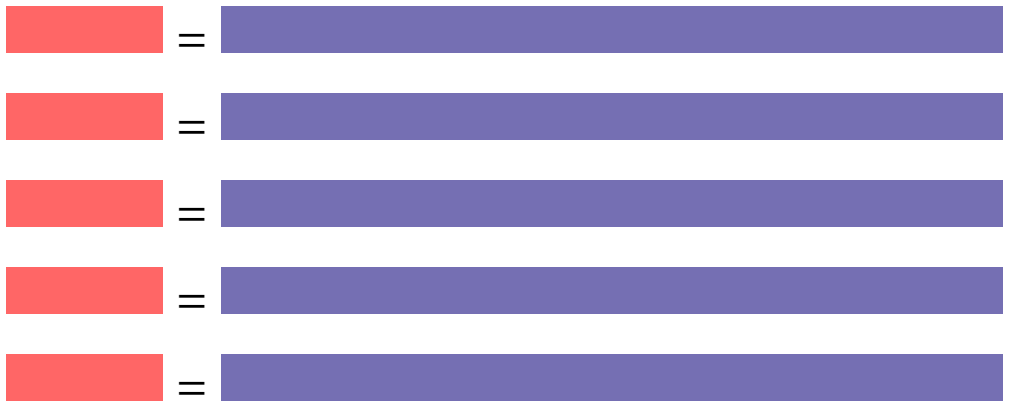
(B) Universal Logic Reasoning in HOL

HOL (meta-logic) φ ::= 

L (object-logic) ψ ::= 

Embedding of  in 



Pass this set of equations to a HOL theorem prover

(B) Universal Logic Reasoning in HOL

HOL $\quad s,t \quad ::= \quad c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML $\quad \varphi,\psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\to o}$ (explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu (\varphi w \wedge \psi w) \\
\to & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu (\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\, h d w \\
\exists & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, h d w \\
\Box & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \forall u_\mu\, (\neg r w u \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\mu\to o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

(B) Universal Logic Reasoning in HOL

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(B) Universal Logic Reasoning in HOL

Example

HOML formula	$\Diamond\exists x Gx$
HOML formula embedded in HOL	$\text{valid}\,(\Diamond\exists x Gx)$
expansion	$(\lambda\varphi\forall w_\mu \varphi\, w)(\Diamond\exists x Gx)$
$\beta\eta$-normalisation	$\forall w_\mu((\Diamond\exists x Gx)\, w)$
expansion	$\forall w_\mu(((\lambda\varphi\lambda w_\mu\exists u_\mu\,(\boldsymbol{r}wu \wedge \varphi u))\exists x Gx)\, w)$
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syntactic sugar	$\forall w_\mu\exists u_\mu(\boldsymbol{r}wu \wedge (\exists(\lambda x Gx))u)$
expansion	$\forall w_\mu\exists u_\mu(\boldsymbol{r}wu \wedge ((\lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu\exists d_\gamma\, hdw)(\lambda x Gx))u)$
$\beta\eta$-normalisation	$\forall w_\mu\exists u_\mu(\boldsymbol{r}wu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid φ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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This can be done with interactive or automated HOL theorem provers.

(B) Universal Logic Reasoning in HOL

Example

HOML formula	$\Diamond\exists x Gx$
HOML formula embedded in HOL	$\text{valid}\,(\Diamond\exists x Gx)$
expansion	$(\lambda\varphi\forall w_\mu \varphi\, w)(\Diamond\exists x Gx)$
$\beta\eta$-normalisation	$\forall w_\mu((\Diamond\exists x Gx)\, w)$
expansion	$\forall w_\mu(((\lambda\varphi\lambda w_\mu \exists u_\mu\, (\boldsymbol{r}wu \wedge \varphi u))\exists x Gx)\, w)$
$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu(\boldsymbol{r}wu \wedge (\exists x Gx)u)$
syntactic sugar	$\forall w_\mu \exists u_\mu(\boldsymbol{r}wu \wedge (\exists(\lambda x Gx))u)$
expansion	$\forall w_\mu \exists u_\mu(\boldsymbol{r}wu \wedge ((\lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, hdw)(\lambda x Gx))u)$
$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu(\boldsymbol{r}wu \wedge \exists x Gxu)$

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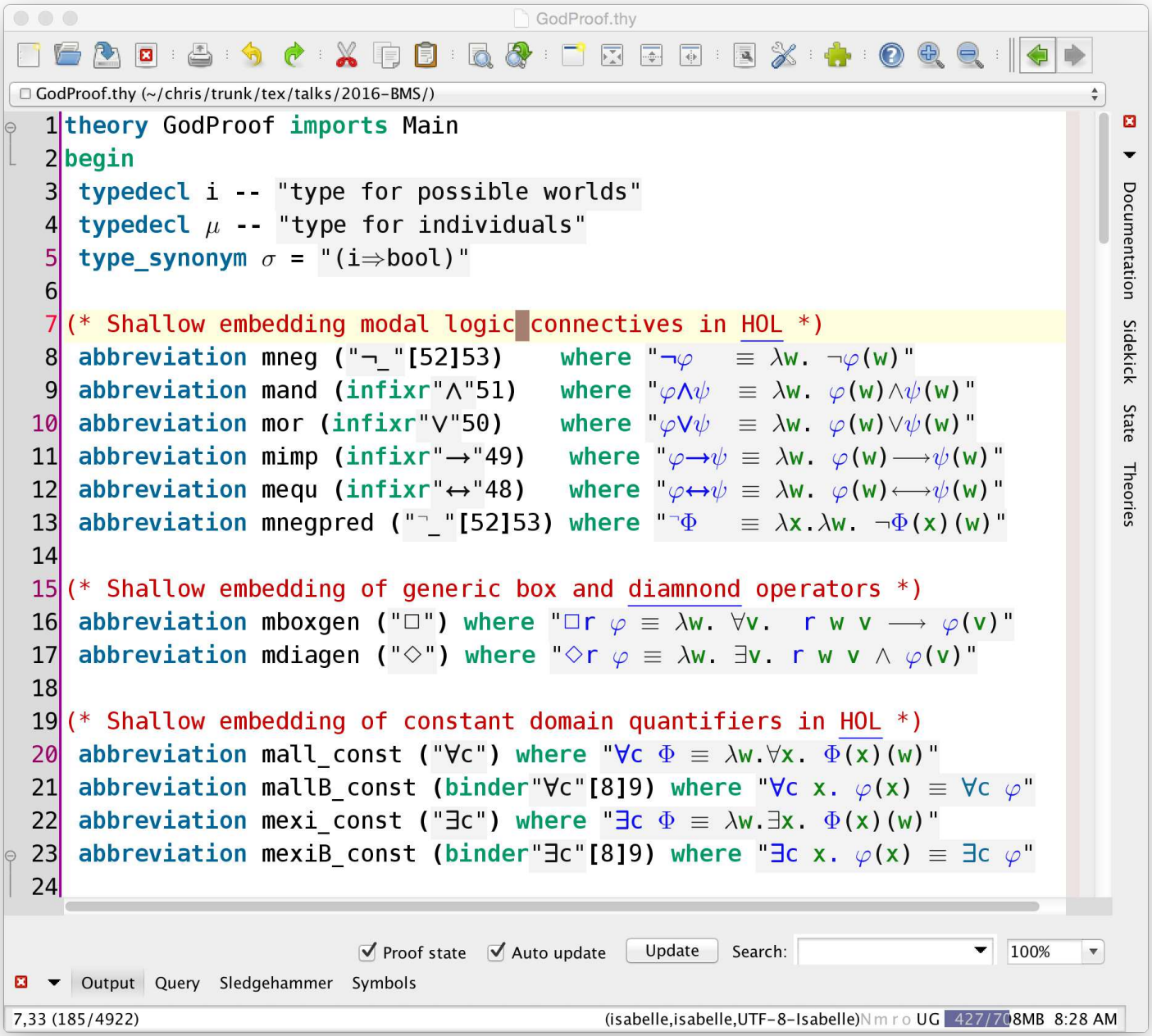
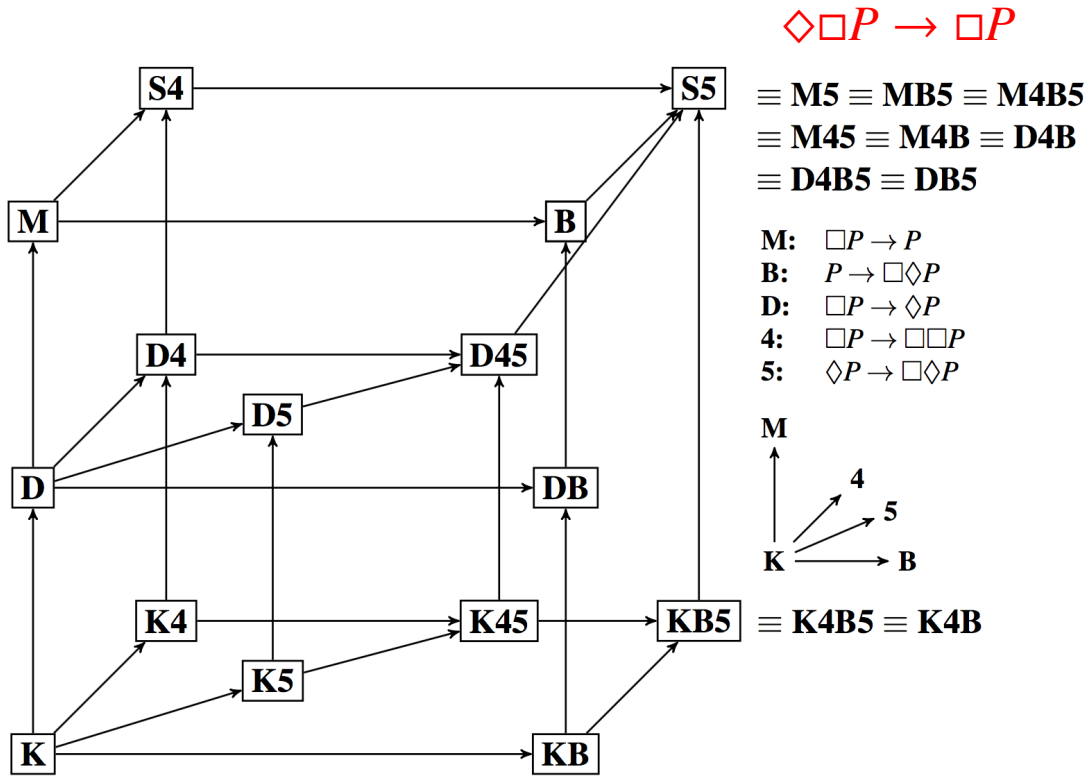
(B) Universal Logic Reasoning in Isabelle/HOL



(B) Universal Logic Reasoning in HOL

Which modal logic?



Which notion of quantification?

- possibilist quantifiers — constant domain semantics
- actualist quantifiers — varying domain semantics

"If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis."

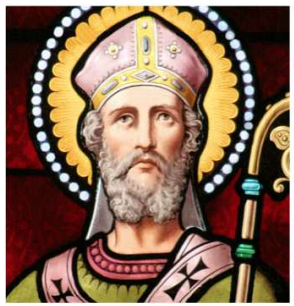
(Leibniz, 1677)

Part C
Computational Metaphysics

jww: Bruno Woltzenlogel-Paleo (ANU, Australia), and others

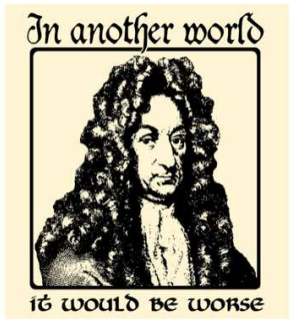
Ontological Proofs of God's Existence

A Long and Continuing Tradition in Philosophy



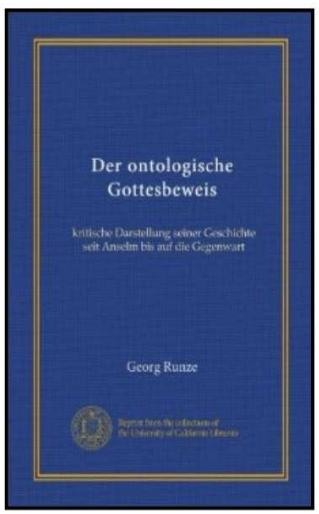
St. Anselm

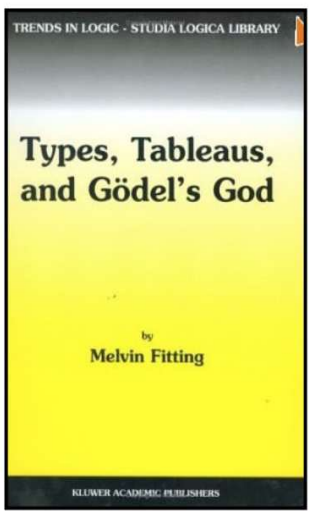
Descartes

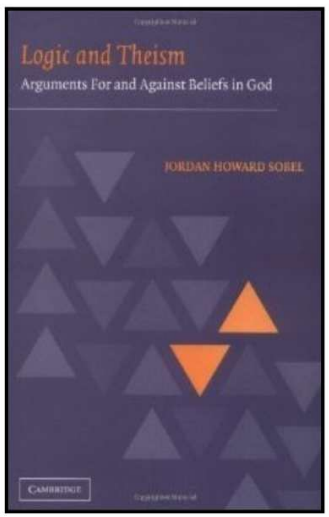


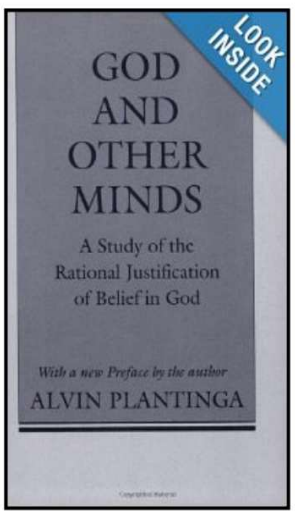
Leibniz

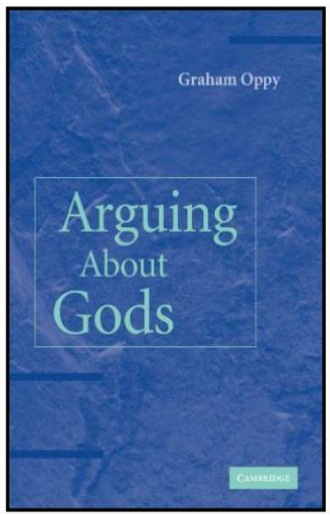
Gödel



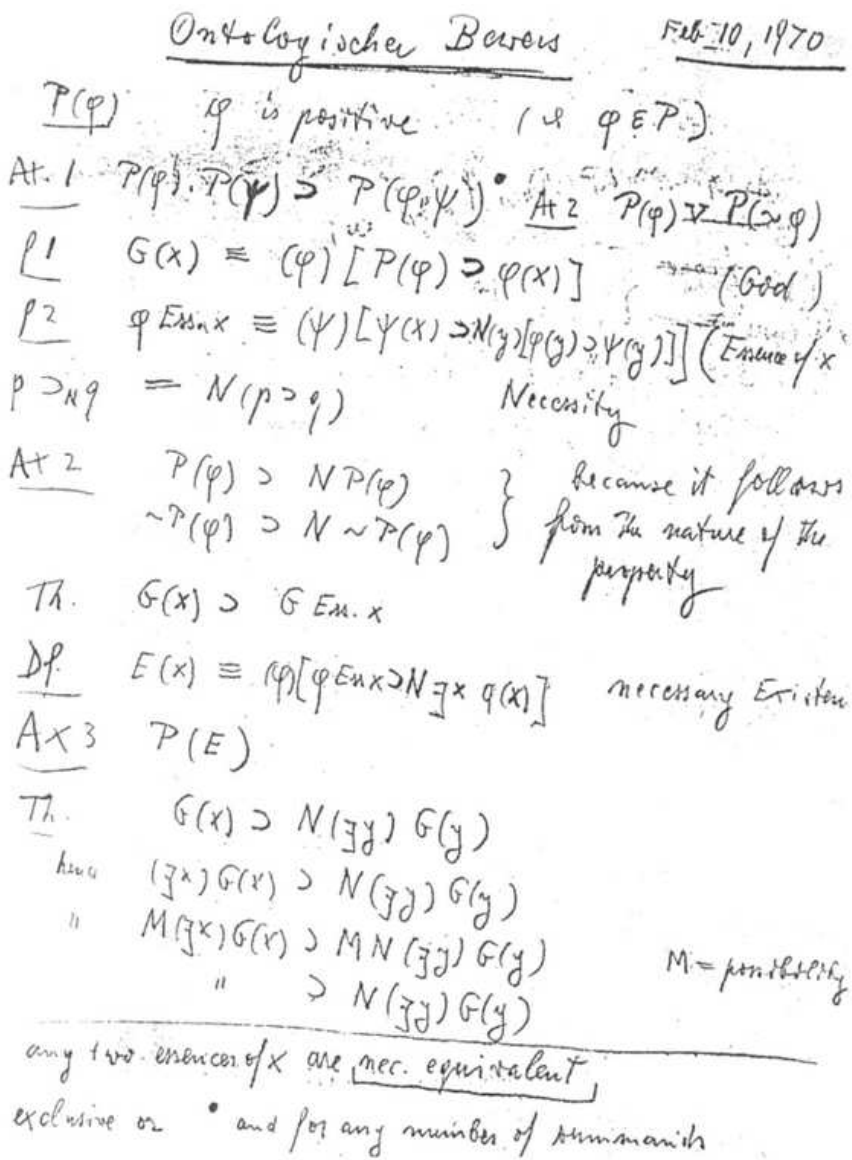








(C) Computational Metaphysics: Kurt Gödel's Ontological Argument



Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ φ is positive ($\varphi \in P$)

Ax. 1 $P(\varphi) \cdot P(\psi) \supset P(\varphi \cdot \psi)$ · Ax 2 $P(\varphi) \vee P(\neg\varphi)$

D1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

D2 $\varphi \, Ess_x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x

$p \supset_N q \;=\; N(p \supset q)$ Necessity

Ax 2 $P(\varphi) \supset N P(\varphi)$ } because it follows
$\neg P(\varphi) \supset N \neg P(\varphi)$ } from the nature of the property

Th. $G(x) \supset G \, Ess. \, x$

Df $E(x) \equiv (\varphi)[\varphi \, Ess \, x \supset N \exists x \, \varphi(x)]$ necessary Existen

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y) G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset MN(\exists y) G(y)$

" $\supset N(\exists y) G(y)$ M = possibility

any two essences of x are nec. equivalent

exclusive or · and for any number of summands

M (∃x) G(x) means "the system of all pos. prop. is com-patible" This is true because of:

Ax 4: $P(\varphi) \cdot \varphi \supset_N \psi : \supset P(\psi)$ which im

~~Thus~~ { $x = x$ is positive
~~end~~ { $x \neq x$ is negative

But if a system S of pos. propo. were in com
it would mean that the sum prop. is (which
is positive) would be $x \neq x$

Positive means positive in the moral aesth
sense (independly of the accidental structure of
the world). Only then the ax. true. But
pure
also means "attribution" as opposed to "privation
(or containg privation) - This interpret. implies proof

If φ positive: (x) $N \neg \varphi(x)$ Otherwise $\varphi(x) \supset_N x \neq$
hence $x \neq x$ positive not $x = x$ by contrary Ax
or the exist. of pos. prop.

dag

x i.e. the normal form in terms of elem. prop. contain
Member without negation.

(C) Computational Metaphysics: Scott's Variant of Gödel's Ontological Argument

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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second-order quantifiers

(C) Computational Metaphysics: Vision of Leibniz (1646–1716) — *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
characteristica universalis and **calculus ratiocinator**

(C) Computational Metaphysics: Scott's and Gödel's Variants — Demo

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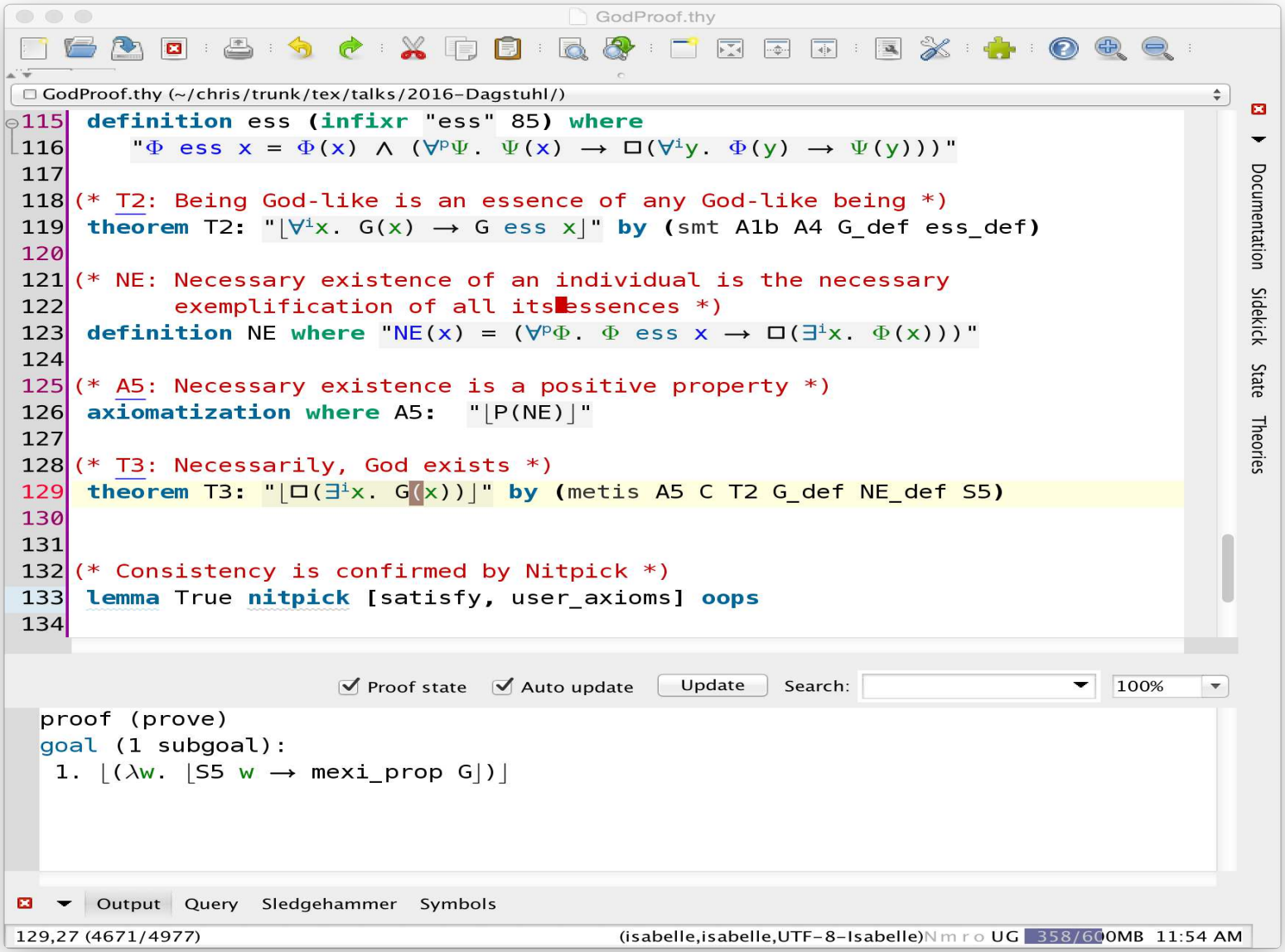

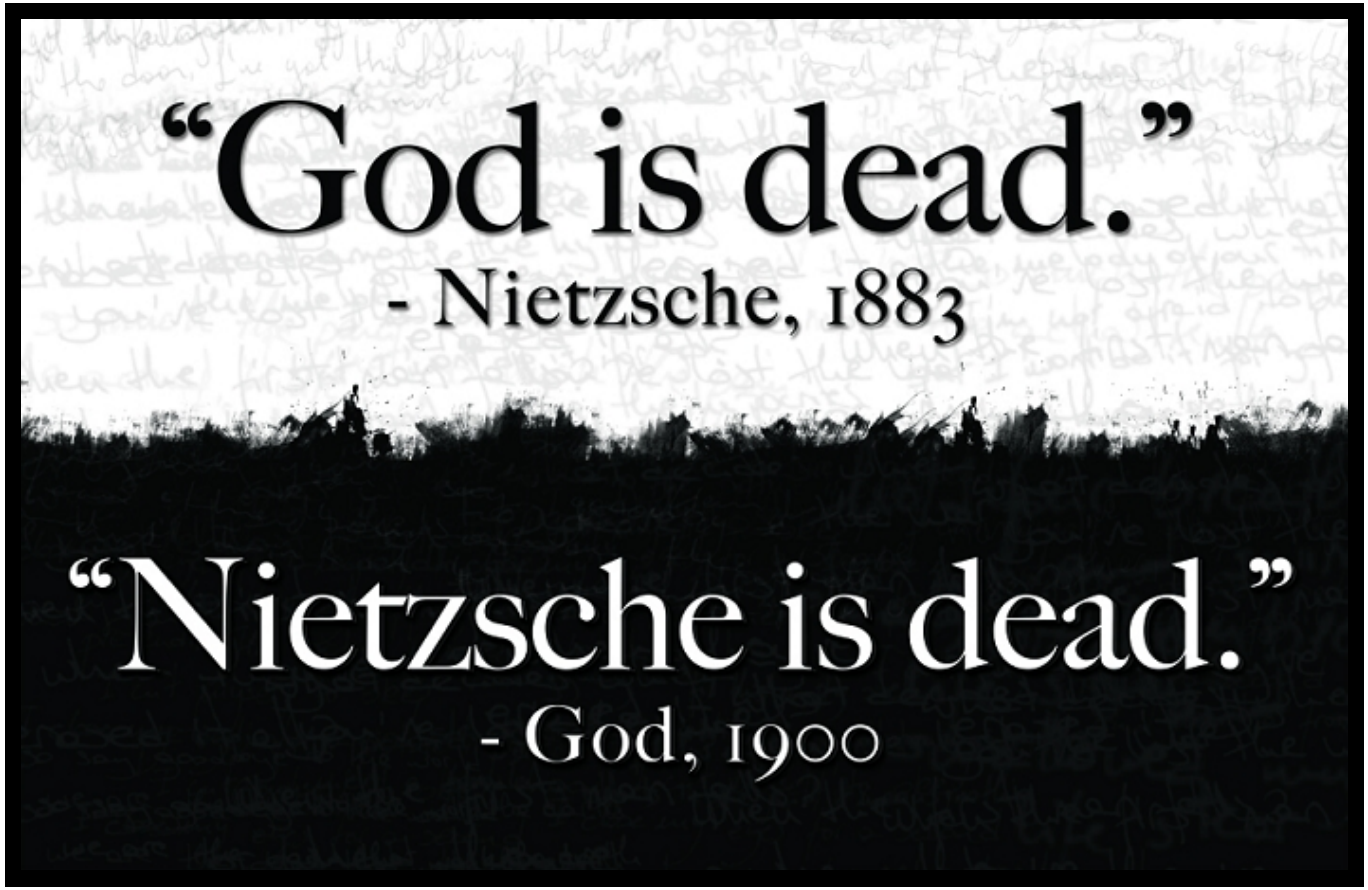

Results of Experiments

see e.g. [ECAI-2014, IJCAI-2016]

(C) Computational Metaphysics: Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathbf{ess}\, \phi X \dot{\supset} \dot{\Box}\dot{\exists}Y_\mu.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$ A1, D1		KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

(C) Computational Metaphysics: Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma} X \dot{\supset} (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma} X]$						
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma} X \dot{\supset} ($						
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma} X \dot{\supset} (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automation of Scott's Variant

Summary

- **Proof verified and automated**
- **Logic KB is sufficient (the often critised S5 not needed!)**
- **Possibilist & actualist quantification (for ind.)**
- **Exact dependencies determined**
- **Theorem provers found alternative Proofs**
  e.g. self-identity $\lambda x(x = x)$ not needed

(C) Computational Metaphysics: Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma} X \dot\supset (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} g_{\mu\to\sigma} X)]$						
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\phi X \dot\supset g_{\mu\to\sigma} X)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Consistency: Gödel vs. Scott

- **Scott's assumptions are consistent; shown by Nitpick**
- **Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result)**

(C) Computational Metaphysics: Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathbf{ess}_{(\mu\to\sigma)}$						
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
		D1, C, T2, D3, A5	K	CSA	—/—	—/—	5.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Further Results

- **Monotheism holds**
- **God is flawless**
- **. . .**

(C) Computational Metaphysics: Results of Experiments

Modal Collapse (detected first by Sobel)

$$\forall \varphi(\varphi \rightarrow \Box\varphi)$$

- proved by LEO-II and Satallax
- already in logic KB
- for possibilist and actualist quantification (ind.)

Modal Collapse can be read as:

- There are no contingent truths
- Everything is determined / there is no free will

	HOL encoding						
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to}$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Diamond}\dot{\exists}$						
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists} X_\mu.\, g_{\mu\to\sigma} X]$						
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Box}p$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$						
T2	$[\dot{\forall} X_\mu.\, g_{\mu\to\sigma} X \,\dot{\supset}\, (\mathbf{ess}_{(\mu\to\sigma)}$						
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists} X_\mu.\, g_{\mu\to\sigma} X]$						
MC	$[s_\sigma \,\dot{\supset}\, \dot{\Box} s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall} X_\mu.\, (g_{\mu\to\sigma} X \,\dot{\supset}\, (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot{\supset}\, \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall} X_\mu.\, \dot{\forall} Y_\mu.\, (g_{\mu\to\sigma} X \,\dot{\supset}\, (g_{\mu\to\sigma} Y \,\dot{\supset}\, X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall} Y_\mu.\, (\phi Y \,\dot{\supset}\, \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

(C) Computational Metaphysics: Avoiding the Modal Collapse

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's Ontological Proof Revisited *

C. Anthony Anderson and Michael Gettings

University of California, Santa Barbara
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

Magari and others on Gödel's ontological proof

Petr Hájek
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1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes
(betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,
sondern glauben, daß ein Gott sei.
(Kant, Nachlaß)

1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

PETR HÁJEK

A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

1. Introduction

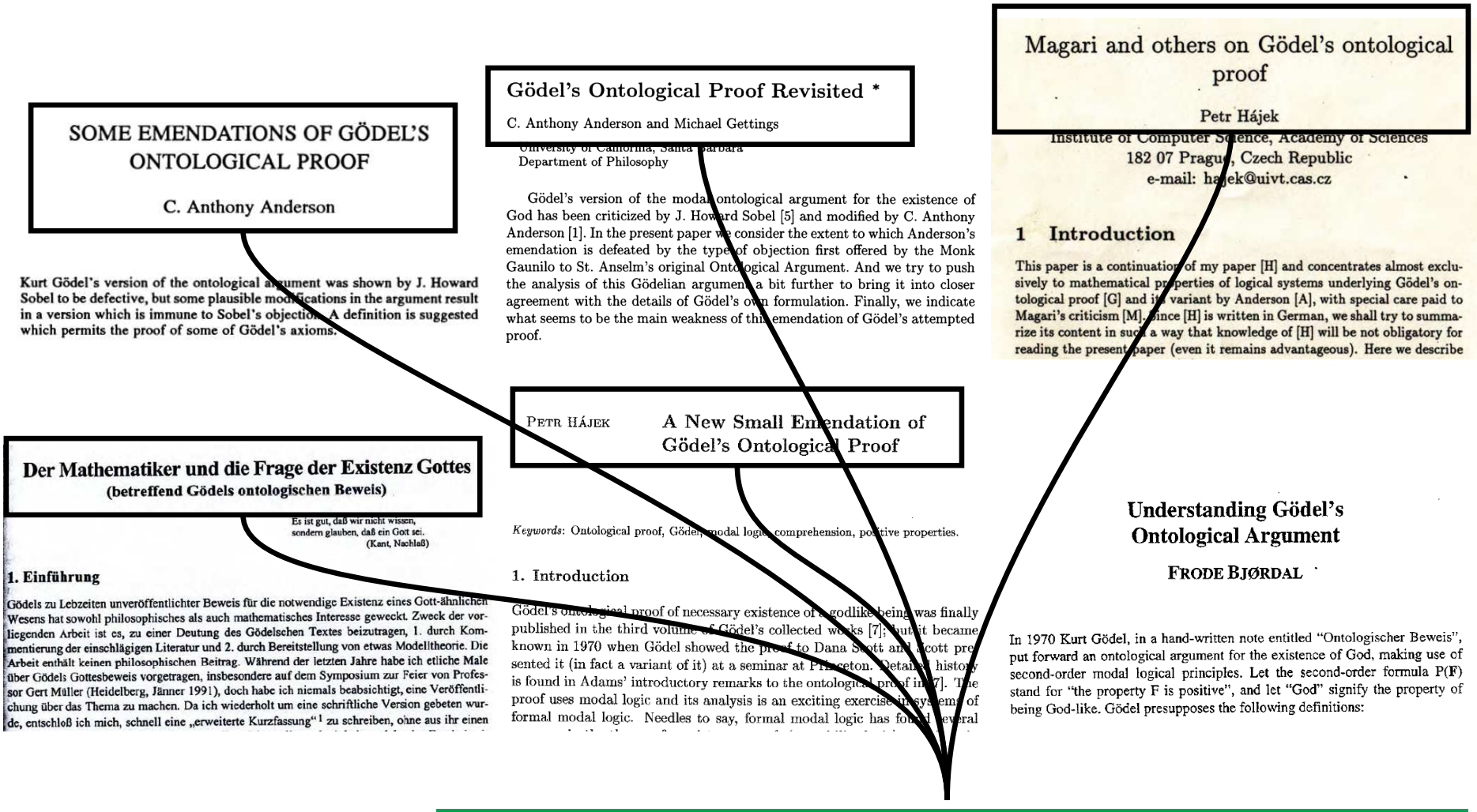
Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

(C) Computational Metaphysics: Avoiding the Modal Collapse



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1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Details of history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

Computer-supported Clarification of Controversy
1st World Congress on Logic and Religion, 2015

(C) Computational Metaphysics: Avoiding the Modal Collapse

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

S/I = superfl. & indep.; R = superfl. & redund.; S/U = superfl. & unknown whether redund. or indep.; N/I = non-superfl. & indep.; P = provable; CS = counter-satisfiable

(C) Computational Metaphysics: Avoiding the Modal Collapse

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

S/I = superfl. & indep.; R = superfl. & redund.; S/U = superfl. & unknown whether redund. or indep.; N/I = non-superfl. & indep.; P = provable; CS = counter-satisfiable

Leibniz (1646–1716)

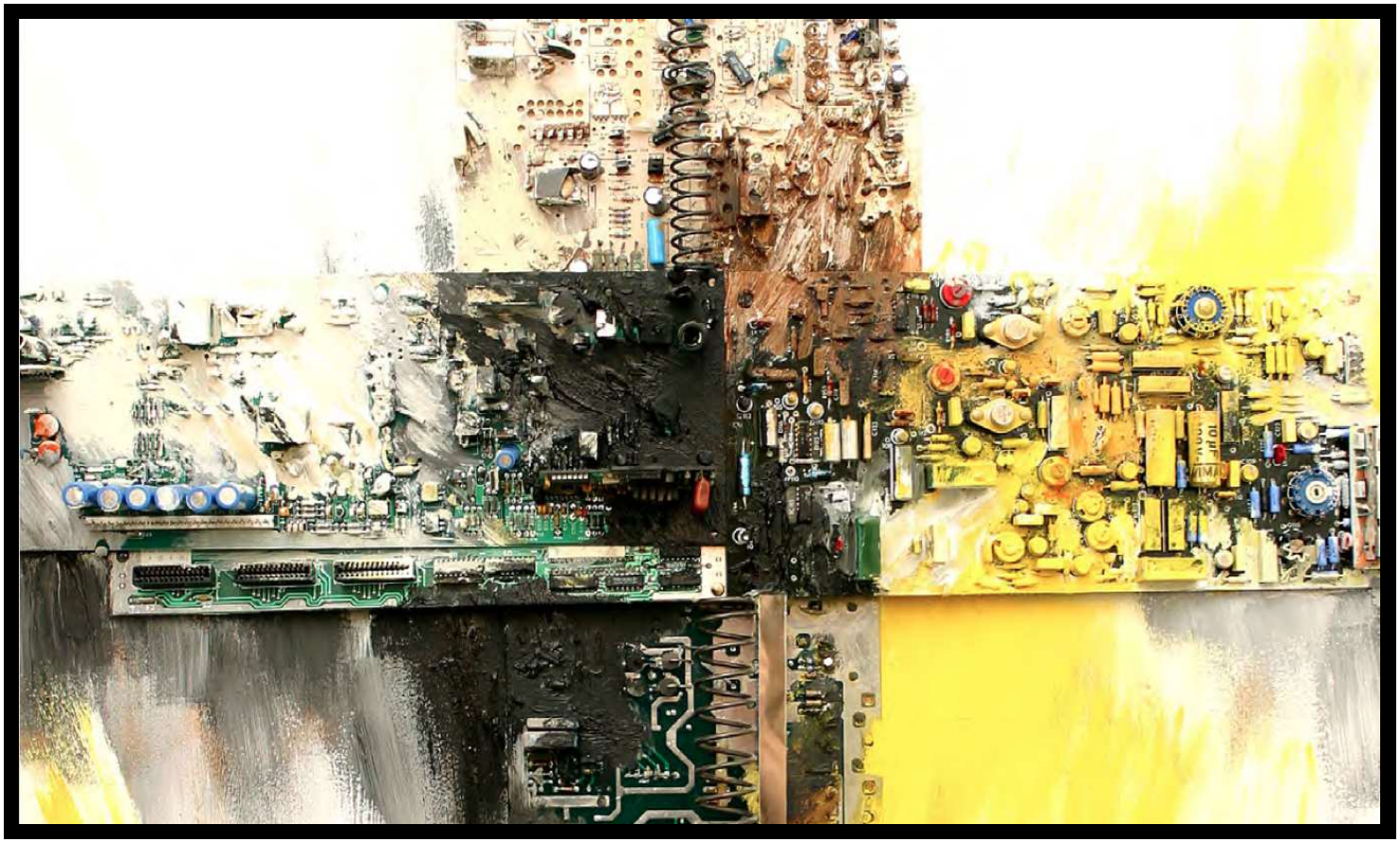
characteristica universalis *and* ***calculus ratiocinator***

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

But: **Intuitive proofs/models are needed to convince philosophers**

(C) Computational Metaphysics: See our Recent Publications

- **Computer-Assisted Analysis of the Anderson-Hájek Controversy**, In Logica Universalis, 2017.
- **Analysis of an Ontological Proof Proposed by Leibniz**, Chapter in Death and Anti-Death, Volume 14, Ria University Press, 2017.
- **An Object-Logic Explanation for the Inconsistency in Gödel's Ontological Theory**, KI 2016, Springer, LNCS, 2016.
- **The Inconsistency in Gödel's Ontological Argument: A Success Story for AI in Metaphysics**, IJCAI 2016, AAAI Press, 2016.
- **The Modal Collapse as a Collapse of the Modal Square of Opposition**, Chapter in The Square of Opposition: A Cornerstone of Thought (Collection of papers related to the World Congress on the Square of Opposition IV, Vatican, 2014), Springer, Studies in Universal Logic, 2016.
- **On Logic Embeddings and Gödel's God**, WADT 2014, Springer, LNCS, 2015.
- **Experiments in Computational Metaphysics: Gödel's Proof of God's Existence**, In Science & Spiritual Quest, 9th All India Students' Conference, Bhaktivedanta Institute, Kolkata, 2015.
- **Invited Talk: On a (Quite) Universal Theorem Proving Approach and Its Application in Metaphysics**, TABLEAUX 2015, Springer, LNAI, 2015.
- **Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers**, ECAI 2014, IOS Press, Frontiers in Artificial Intelligence and Applications, 2014.
- ...
- Further papers in preparation



Rational Reconstruction of the Inconsistency of Gödel's Axioms

[Benzmüller&WoltzenlogelPaleo, IJCAI-2016]

(C) Computational Metaphysics: Rational Reconstruction of Inconsistency

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists x G(x)$

Difference to Gödel (who omits this conjunct)

(C) Computational Metaphysics: Rational Reconstruction of Inconsistency

(C) Computational Metaphysics: Rational Reconstruction of Inconsistency

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

Def. D2* $\phi\ ess.\ x \leftrightarrow \cancel{\phi(x)} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset\ ess.\ x)$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Axiom A5 $P(NE)$

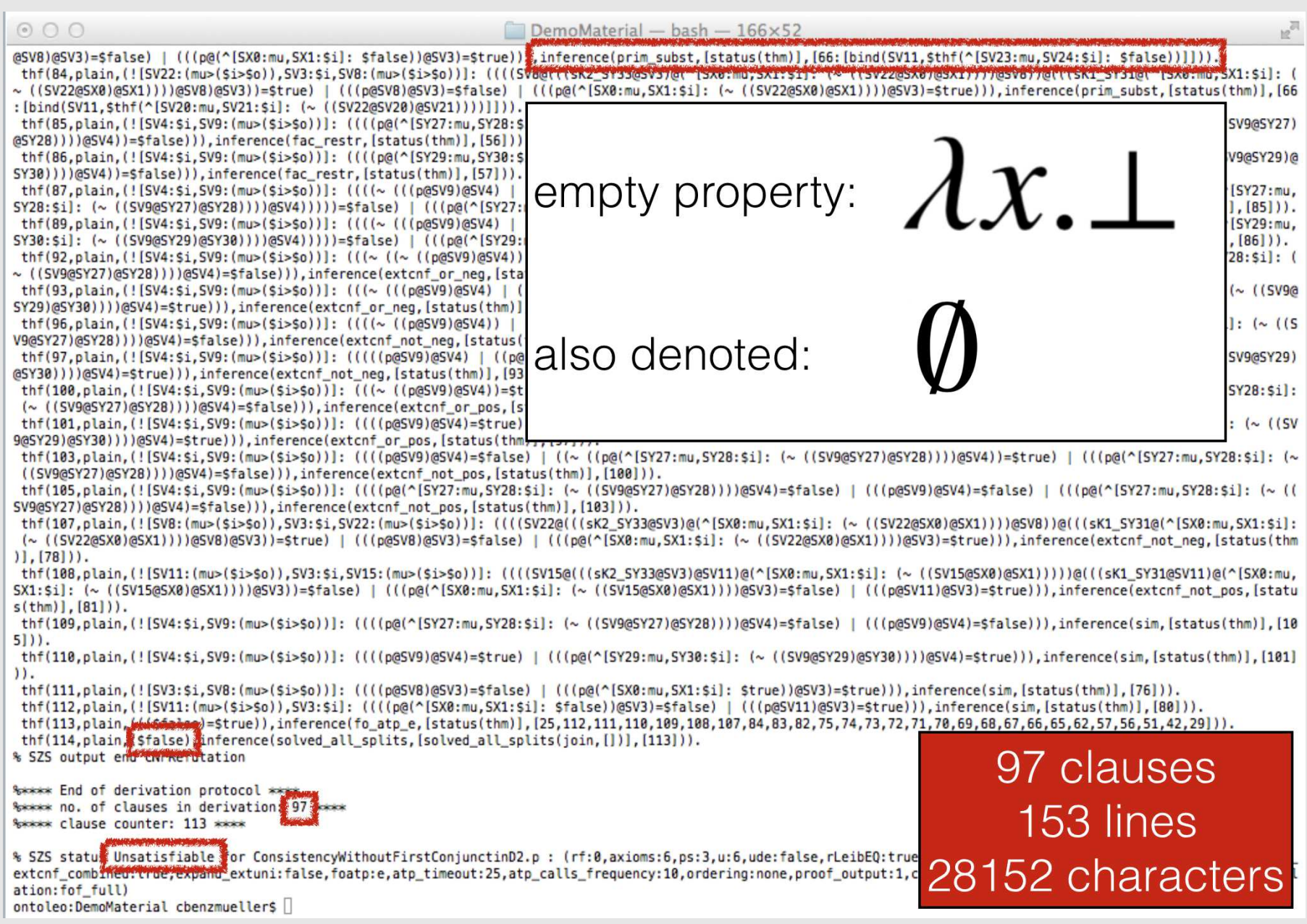
- by T1, A5: $\Diamond\exists x[NE(x)]$

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

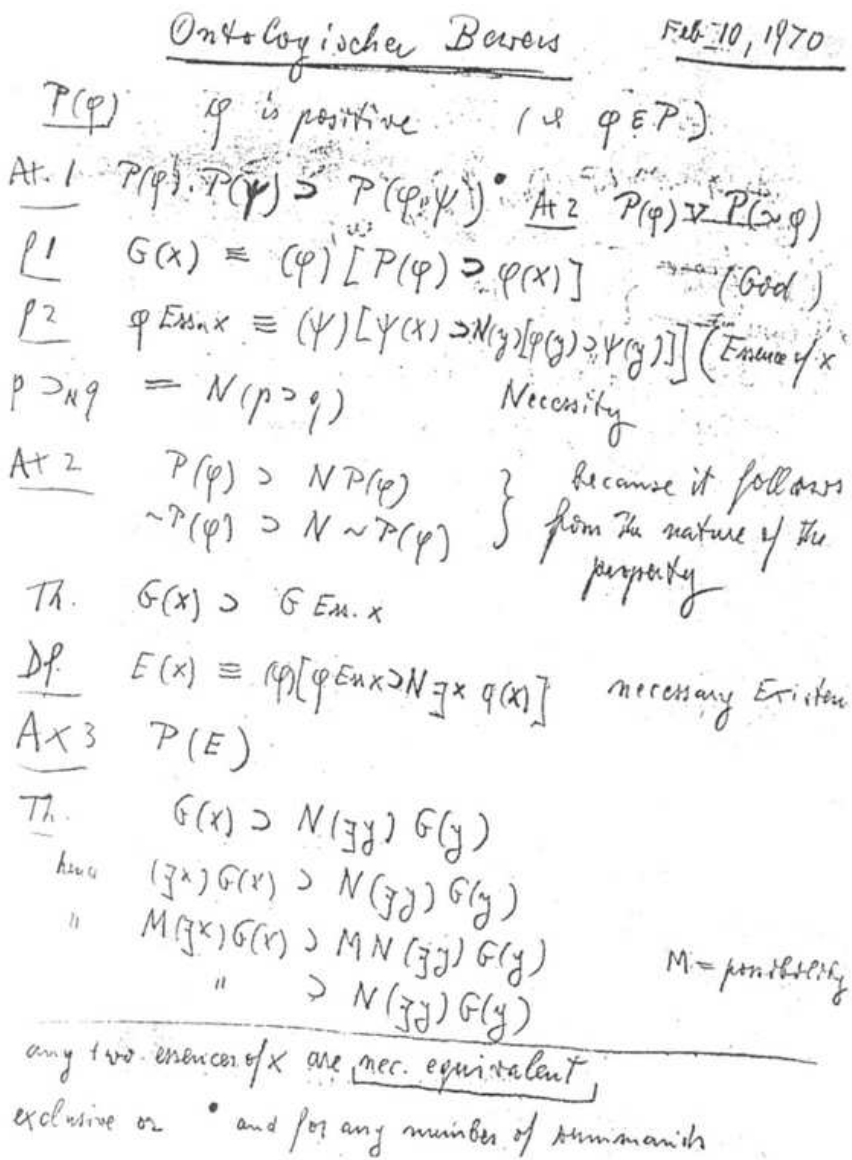
- by D3 $\Diamond\exists x[\forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$
- $\Diamond\exists x[\emptyset\ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$
- by Lemma 1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$
- by def. of $\emptyset$ $\Diamond\exists x[\top \rightarrow \Box\bot]$
- $\Diamond\exists x[\Box\bot]$
- $\Diamond\Box\bot$

Inconsistency $\bot$

(C) Computational Metaphysics: Rational Reconstruction of Inconsistency



(C) Computational Metaphysics: Gödel's Manuscript — Inconsistent Axioms



Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ φ is positive ($\varphi \in P$)

Ax. 1 $P(\varphi) \cdot P(\psi) \supset P(\varphi \cdot \psi)$ · Ax 2 $P(\varphi) \vee P(\neg\varphi)$

D1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

D2 $\varphi \, Ess.x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x

$p \supset_N q \;=\; N(p \supset q)$ Necessity

Ax 2 $P(\varphi) \supset N P(\varphi)$ } because it follows from the nature of the property
$\neg P(\varphi) \supset N \neg P(\varphi)$ }

Th. $G(x) \supset G \, Ess. x$

Df $E(x) \equiv (\varphi)[\varphi \, Ess \, x \supset N \exists x \, \varphi(x)]$ necessary Existen

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y) G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset M N (\exists y) G(y)$ M = possibility

" $\supset N(\exists y) G(y)$

any two essences of x are nec. equivalent

exclusive or · and for any number of summands

$M(\exists x) G(x)$ means "the system of all pos. prop. is com-patible" This is true because of:

Ax 4: $P(\varphi) \cdot \varphi \supset_N \psi : \supset P(\psi)$ which im

~~Lemma~~ { $x = x$ is positive
~~end~~ { $x \neq x$ is negative

But if a system S of pos. propo. were in com
it would mean that the sum prop. is (which
is positive) would be $x \neq x$

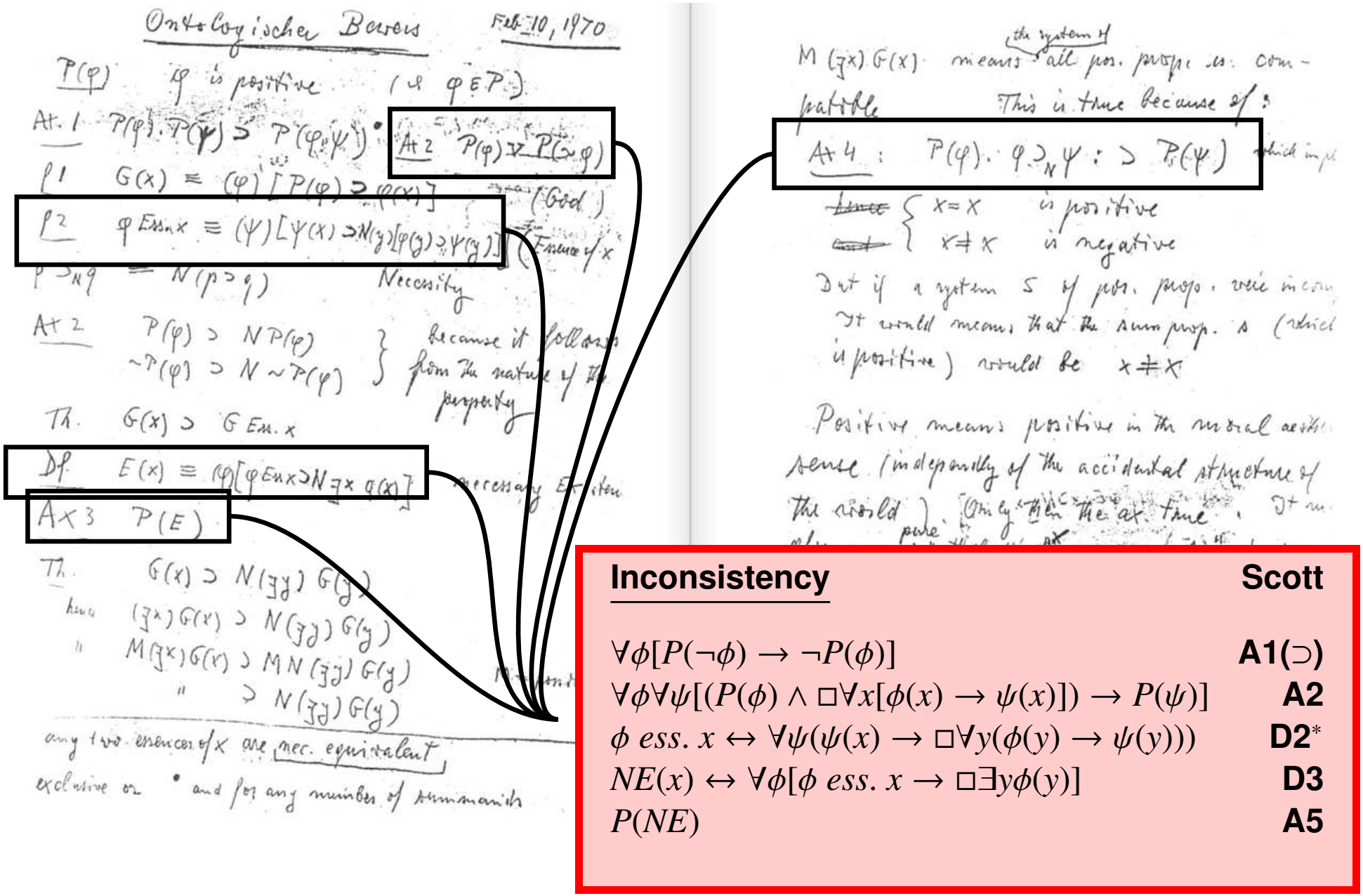
Positive means positive in the moral aesth
sense (independly of the accidental structure of
the world). Only then the ax. true. But
also meant "attribution" as opposed to "privation
(or containg privation) - This interpret. to later proof

If φ positive: $(x) N \neg \varphi(x)$ Otherwise $\varphi(x) \supset_N x \neq$
hence $x \neq x$ positive not $x = x$ by contrary Ax
or the existence of pos. prop.

dag

x i.e. the normal form in terms of elem. prop. contain
Member without negation.

(C) Computational Metaphysics: Gödel's Manuscript — Inconsistent Axioms



Inconsistency — **Scott**

$\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$	**A1(⊃)**
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$	**A2**
$\phi\ ess.\ x \leftrightarrow \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$	**D2***
$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$	**D3**
$P(NE)$	**A5**

Computational Metaphysics
— Recent and Ongoing Work —

jww: Ed Zalta (Stanford, US), Dana Scott (Berkeley, US), Bruno Woltzenlogel Paleo (Canberra, Australia), Alex Steen and Max Wisniewski (Berlin), many others

(C) Computational Metaphysics: Lecture Course on Computational Metaphysics

Metaphysics: *Foundational Branch in Philosophy*, that ...

... studies the fundamental nature of **being** and **the world** that encompasses it

... looks **beyond experience** in the real world

Flammarions Holzstich – in L'atmosphère, Paris 1888

Adresses **ultimate questions**, such as:

- What is there?
- What is it like?
- **Is there a God?**
- What can I know?

Method: **Rational Argumentation**

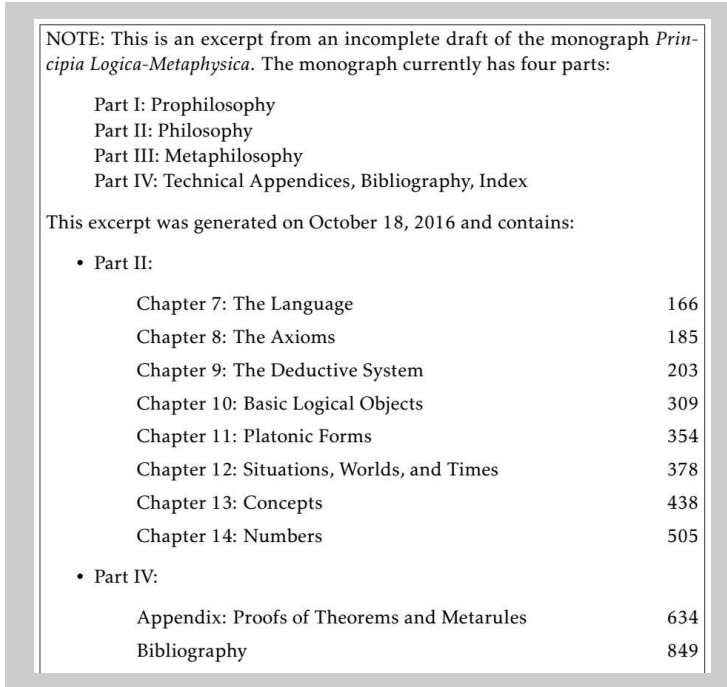
Lecture Course won the 2015 Central Teaching Award of FU Berlin

(jww: Alex Steen, Max Wisniewski, and others)

- MSc students in Maths, CS, Philosophy and Physics from FU, TU and HU
- Invited Lectures by Philosophers (Zalta & Lenzen) and Computer Scientists
- First course of this kind worldwide!

(C) Computational Metaphysics: Principia Metaphysica (Zalta)

–since ’83→



NOTE: This is an excerpt from an incomplete draft of the monograph *Principia Logica-Metaphysica*. The monograph currently has four parts:

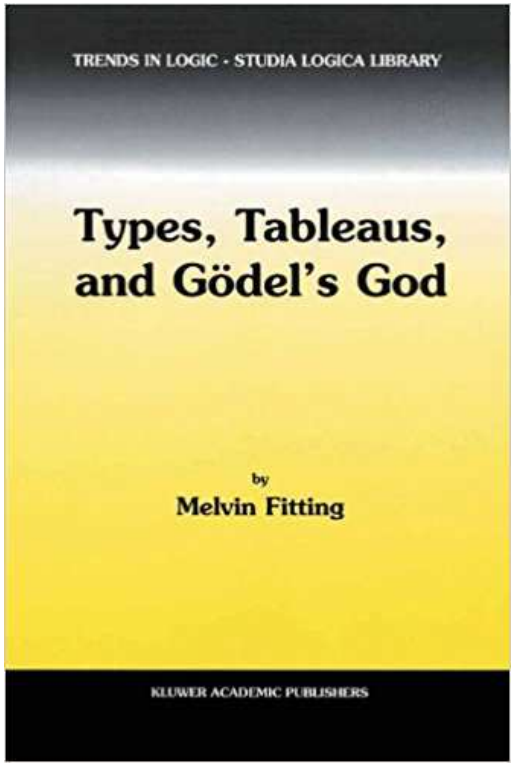
Part I: Prophilosophy
Part II: Philosophy
Part III: Metaphilosophy
Part IV: Technical Appendices, Bibliography, Index

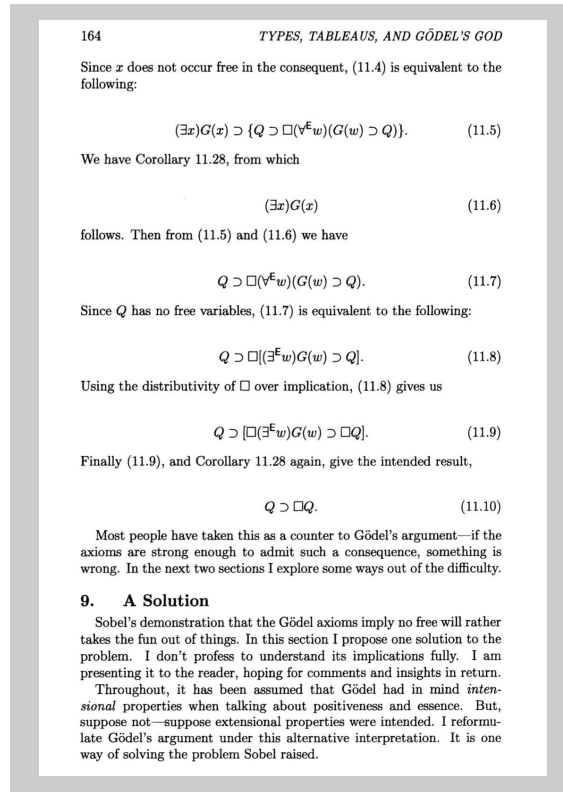
This excerpt was generated on October 18, 2016 and contains:

Principia Logico-Metaphysica

- Objective: Formalisation/Automation of Principia Metaphysica
- Joint work with: Daniel Kirchner (student of mine), Ed Zalta (Stanford)
- Logics: Hyper-Intensional Higher-Order Modal Logic
- Results:
  - PLM fully formalised (by Daniel), only minor issues detected
  - good degree of automation, abstract level theorem prover
- Sources: `https://github.com/ekpyron/TAO`

(C) Computational Metaphysics: Intensional Version of Ontological Argument





164 *TYPES, TABLEAUS, AND GÖDEL'S GOD*

Since x does not occur free in the consequent, (11.4) is equivalent to the following:

$$(\exists x)G(x) \supset \{Q \supset \Box(\forall^{\mathsf{E}} w)(G(w) \supset Q)\}. \quad (11.5)$$

We have Corollary 11.28, from which

$$(\exists x)G(x) \quad (11.6)$$

follows. Then from (11.5) and (11.6) we have

$$Q \supset \Box(\forall^{\mathsf{E}} w)(G(w) \supset Q). \quad (11.7)$$

Since Q has no free variables, (11.7) is equivalent to the following:

$$Q \supset \Box[(\exists^{\mathsf{E}} w)G(w) \supset Q]. \quad (11.8)$$

Using the distributivity of $\Box$ over implication, (11.8) gives us

$$Q \supset [\Box(\exists^{\mathsf{E}} w)G(w) \supset \Box Q]. \quad (11.9)$$

Finally (11.9), and Corollary 11.28 again, give the intended result,

$$Q \supset \Box Q. \quad (11.10)$$

Most people have taken this as a counter to Gödel's argument—if the axioms are strong enough to admit such a consequence, something is wrong. In the next two sections I explore some ways out of the difficulty.

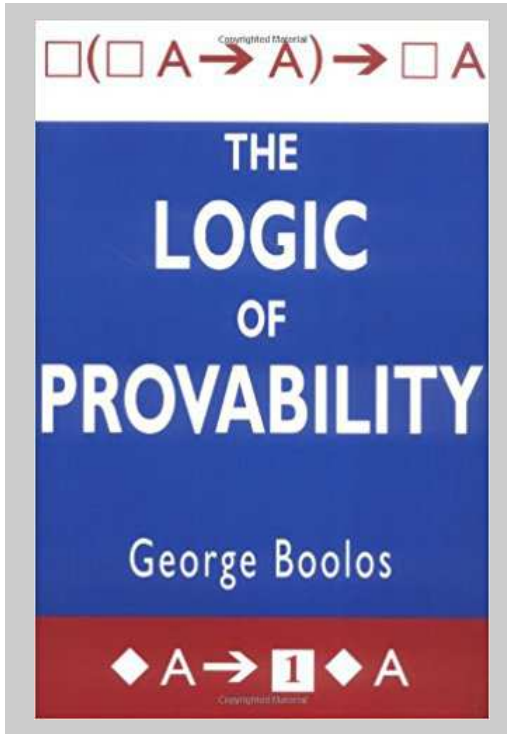
9. A Solution

Sobel's demonstration that the Gödel axioms imply no free will rather takes the fun out of things. In this section I propose one solution to the problem. I don't profess to understand its implications fully. I am presenting it to the reader, hoping for comments and insights in return.

Throughout, it has been assumed that Gödel had in mind *intensional* properties when talking about positiveness and essence. But, suppose not—suppose extensional properties were intended. I reformulate Gödel's argument under this alternative interpretation. It is one way of solving the problem Sobel raised.

- Objective: Formalisation/Automation of Fitting's Book
- Joint work with: David Fuenmayor (student of mine)
- Logics: Intensional Higher-Order Modal Logic
- Results:
  - essential parts are formalised (by David), only minor issues detected so far
  - good degree of automation
- Sources: `https://github.com/cbenzmueller/TypesTableauxAndGoedelsGod`

(C) Computational Metaphysics: Provability Logic



- Objective: Formalisation/Automation of Boolos' Book
- Joint work with: David Streit (student of mine)
- Logics: Provability Logic
- Results:
  - essential parts are already formalised, no issues detected so far
  - good degree of automation
- Sources: `will-soon-be-made-public`

Conclusion

- Formal Proofs Increasingly Relevant in Maths, CS and Beyond
- Significant Improvements in Interactive and Automated Theorem Proving
- Universal Logic Reasoning in HOL via Shallow Semantic Embeddings
- **Deep Analysis of Rational Arguments on the Computer**
  - significant contribution towards a **Computational Metaphysics**
  - even **novel results** contributed by **theorem provers**
  - related work: only for Anselm's simpler argument
    - first-order ATP PROVER9 [OppenheimerZalta, 2011]
    - interactive proof assistant PVS [Rushby, 2013]
  - **approach works well in practice**: matches granularity of human arguments
  - **approach can be adapted** for other logics and logic combinations

Outlook: Many relevant other applications (but I need resources!)

- Maths, Computer Science, Artificial Intelligence
- Philosophy, Natural Language Processing
- Legal-, Ethical- and Moral-Reasoning in Intelligent Machines
- Rational Argumentation in general