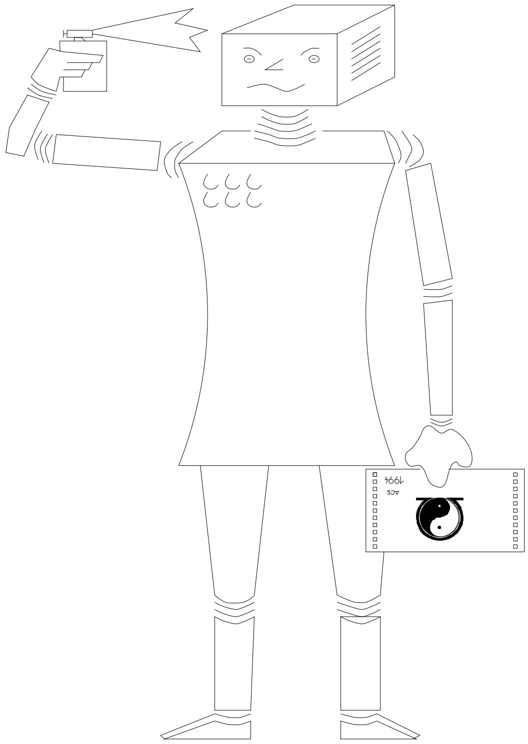

SEKI Report

Model Existence for Higher Order Logic

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Abstract

In this paper we provide a semantical meta-theory that will support the development of higher-order calculi for automated theorem proving like the corresponding methodology has in first-order logic. To reach this goal, we establish classes of models that adequately characterize the existing theorem-proving calculi and we present a standard methodology of abstract consistency methods (by providing the necessary model existence theorems) needed to analyze completeness of machine-oriented calculi with respect to this model classes.

We further parameterize the introduced semantical structures and the corresponding abstract consistency properties with an order k. This provides a finer proof methodology which can be used to show that the primitive substitution rule in resolution calculi as discussed in [And71] or [BK97] can be restricted with respect to the order of the input problems without losing completeness.

Contents

1 Motivation

In classical first-order predicate logic, it is rather simple to assess the deductive power of a calculus: first-order logic has a well-established and intuitive set-theoretic semantics, relative to which, completeness can easily be verified using for instance the abstract consistency method (see for instance the introductory textbooks [And86, Fit90]). This well-understood meta-theory has supported the development of calculi adapted to special applications – such as automated theorem proving (see for instance [Bib98] for an overview).

In higher-order logics, the situation is rather different: the intuitive set-theoretic standard semantics cannot give a sensible notion of completeness, since it does not admit complete calculi [Göd31]. However, there is a more general notion of semantics (the so-called Henkin-models [Hen50]), that allows complete calculi and therefore sets the standard for deductive power of calculi.

Peter Andrews' "Unifying Principle for Type Theory" [And71] provides a method of higher-order abstract consistency that has become the standard tool for completeness proofs in higher-order logic, even though it can only be used to show completeness relative to a certain Hilbert style calculus $\mathfrak{T}$. A calculus $\mathcal{C}$ is called complete relative to a calculus $\mathcal{C}'$, iff $\mathcal{C}$ proves all theorems of $\mathcal{C}'$. Since $\mathfrak{T}$ is not complete with respect to Henkin models, the notion of completeness that can be established by this method is a strictly weaker notion than Henkin completeness.

As a consequence, the calculi developed for higher-order automated theorem proving [Hue73, And71, Mil83, Koh95] and the corresponding theorem proving systems such as TPS [ABI^{+}96], or earlier versions of the authors' LEO[1] are not or cannot be proven complete with respect to Henkin models. Moreover, they are not even sound with respect to $\mathfrak{T}$, since all of them utilize η-conversion, which is not a theorem of $\mathfrak{T}$. In other words, their deductive power lies somewhere between $\mathfrak{T}$ and Henkin models.

In this situation, the aim of this paper is to provide a semantical meta-theory that will support the development of higher-order calculi for automated theorem proving like the corresponding methodology has in first-order logic. To reach this goal, we establish

- classes of models that adequately characterize the deductive power of existing theorem-proving calculi (making them sound and complete), and
- a standard methodology of abstract consistency methods (by providing the necessary model existence theorems, which extend Andrews' Unifying Principle), so that the completeness analysis for higher-order calculi will become a simple exercise like in first-order logic.

Due to the inherent complexity of higher-order semantics we will use the next section for an informal exposition of the issues covered and the techniques applied.

2 Informal Exposition

Before we turn to the exposition of the semantics in section 2.2 and discuss some applications of the results (section 2.3), let us specify what we mean by "higher-order logic": any simply typed logical system that allows quantification over function and predicate variables. Technically, we will employ a logical system $\mathcal{HOL}$, which is based on the simply typed λ-calculus. A related logical system is discussed in detail in [And86].

2.1 Higher-Order Logic ($\mathcal{HOL}$)

$\mathcal{HOL}$-formulae are built up from the set $\mathcal{V}$ of variables, and the **signature** Σ (a set of typed constants) as **applications** and **λ-abstractions.** The set $\mathit{wff}_\alpha(\Sigma)$ of **well-formed formulae** term consists of those that can be given a type α so that in all applications, the types of the

[1]Based on the results of this paper the resolution calculus underlying newer versions of LEO [Ben97, BK98] can be proven complete for Henkin models based, see [BK97].

arguments are the argument types of the function. We will denote variables with upper-case letters ($X_\alpha, Y, Z, X^1_\beta, X^2_\gamma \ldots$), constants with lower-case letters ($c_\alpha, f_{\alpha\to\beta}, \ldots$) and well-formed formulae with upper-case bold letters ($\mathbf{A}_\alpha, \mathbf{B}, \mathbf{C}^1, \ldots$)[2]. Finally, we abbreviate multiple applications and abstractions in a kind of vector notation, so that $\mathbf{A}\overline{\mathbf{U}^k}$ denotes k-fold application (associating to the left) and $\lambda\overline{X^k}.\mathbf{A}$ denotes k-fold λ-abstraction (associating to the right) and use the square dot **.** as an abbreviation for a pair of brackets, where **.** stands for the left one with its partner as far to the right, as is consistent with the bracketing already present in the formula.

We will use the terms like **free** and **bound** variables or **closed** formulae in their standard meaning and use $\mathbf{Free}(\mathbf{A})$ for the set of free variables of a formula $\mathbf{A}$. In particular alphabetic change of names of bound variables as built into our $\mathcal{HOL}$: we consider alphabetic variants to be identical (viewing the actual representation as a representative of an alphabetic equivalence class) and use a notion of substitution that avoids variable capture, systematically renaming bound variables. We could also have used de Bruijn's indices [dB72], as a concrete implementation of this approach at the syntax level.

We denote a substitution that instantiates a free variable X with a formula $\mathbf{A}$ with $[\mathbf{A}/X]$ and write $\sigma, [\mathbf{A}/X]$ for the substitution that is identical with σ but instantiates X with $\mathbf{A}$.

If $\mathbf{A}$ has a subterm $\mathbf{B}$ at position p, we denote this by $\mathbf{A}[\mathbf{B}]_p$ and we will write the operation of replacing this subterm by a formula $\mathbf{C}$ with $[\mathbf{C}/p]\mathbf{A}$.

The structural equality relation of $\mathcal{HOL}$ is induced by **$\beta\eta$-reduction**

$$(\lambda X.\mathbf{A})\mathbf{B} \longrightarrow_\beta [\mathbf{B}/X]\mathbf{A} \qquad\qquad (\lambda X.\mathbf{C}X) \longrightarrow_\eta \mathbf{C}$$

where X is not free in $\mathbf{C}$. It is well-known, that the reduction relations β, η, and $\beta\eta$ are terminating and confluent on $\mathit{wff}_\alpha(\Sigma)$, so that there are unique $\beta\eta$ normal forms (see for instance [Bar84] for an introduction).

In $\mathcal{HOL}$, the set of base types is $\{o, \iota\}$ for truth values and individuals. We will call a formula of type o a **proposition** and a **sentence**, if it is closed. We will assume that the signature Σcontains logical constants for **negation** $\neg_{o\to o}$, **conjunction** $\wedge_{o\to o\to o}$, **quantification**[3] $\Pi^\alpha_{(\alpha\to o)\to o}$, and possibly **equality** $=^\alpha_{\alpha\to\alpha\to o}$, all other constants are called **parameters**, since the argumentation in this paper is parametric their choice; we only assume that there are closed formulae for both base types, and as a consequence that all types are non-empty. In particular, we do not assume the existence of description or choice operators. For a detailed discussion of the semantic issues raised by the presence of these logical constants see [And72b].

It is matter of folklore that equality can directly be expressed in $\mathcal{HOL}$ e.g. by the **Leibniz formula** for equality in terms of the other connectives

$$\mathbf{Q}^\alpha := (\lambda X_\alpha Y_\alpha.\forall P_{\alpha\to o}.PX \Rightarrow PY)$$

With this definition, the formula ($\mathbf{A} \doteq^\alpha \mathbf{B}$), which we use as an abbreviation for $\mathbf{Q}^\alpha\mathbf{A}\mathbf{B}$ β-reduces to $\forall P_{\alpha\to o}.(P\mathbf{A}) \Rightarrow (P\mathbf{B})$, which can be read as: formulae $\mathbf{A}$ and $\mathbf{B}$ are not equal, iff there exists a discerning property P[4]. In other words, $\mathbf{A}$ and $\mathbf{B}$ are equal, if they are indiscernible. There are alternatives to define equality in terms of the logical connectives (see for example [And86, p. 155]).

In this paper we differentiate between five different notions of equality. In order to prevent misunderstandings we explain these different notions together with their syntactical representation here:

If we *define* a concept we use $:=$ (e.g. let $\mathcal{D} := \{\mathtt{T}, \mathtt{F}\}$) and $\equiv$ represents *Meta-equality.* We refer to the equality relation as an *object* of our *semantical domains* with q; note that we possibly have one q^α in each domain $\mathcal{D}_\alpha$. The remaining two notions, $\doteq$ and $=$, are related to syntax. $=^\alpha$ may occur as a *constant symbol* of type α in a signature Σ and finally $\doteq^\alpha$ (and sometimes also $\mathbf{Q}^\alpha$) for *Leibniz equality.*

[2] We will denote the type of formulae as an index, if it is not clear from the context.

[3] With this quantification constant, standard quantification of the form $\forall X_\alpha.\mathbf{A}$ can be regained as an abbreviation for $\Pi^\alpha(\lambda X_\alpha.\mathbf{A})$

[4] Note that by contraposition we easily get the backward direction of $\Leftarrow$ and hence it is sufficient to use $\Rightarrow$ instead of $\Leftrightarrow$.

2.2 Notions of Models for $\mathcal{HOL}$

Let us now explore the semantic notions needed to understand figure 1. We will discuss the model classes from bottom to top, from the most specific notion of standard models ($\mathfrak{ST}$)to the most general notion of υ-complexes, motivating the respective generalizations as we go along. In section 3, we will proceed the other way around, specializing the notion of a Σ-model ($\mathfrak{M}$) more and more.

The symbols in the boxes in figure 1 denote model classes, the symbols labeling the arrows indicate the properties inducing the corresponding specialization, and the ∇-symbols next to the boxes indicate the clauses in the definition of abstract consistency class (cf. 4.4) that are needed to establish a model existence theorem for this class of models.

A **standard model** ($\mathfrak{ST}$, cf. Definition 3.30) for $\mathcal{HOL}$ provides a fixed set $\mathcal{D}_\iota$ of individuals, and a set $\mathcal{D}_o := \{\mathtt{T},\mathtt{F}\}$ of truth values. All the domains for the complex types are defined inductively: $\mathcal{D}_{\alpha\to\beta}$ is the set of functions $f\colon\mathcal{D}_\alpha \to \mathcal{D}_\beta$. The evaluation function $\mathcal{I}_\varphi$ with respect to an interpretation $\mathcal{I}\colon\Sigma \to \mathcal{D}$ of constants and an assignment φ of variables is obtained by the standard homomorphic construction that evaluates a λ-abstraction with a function, whose operational semantics is specified by β-reduction.

One can reconstruct the key idea behind **Henkin models** ($\mathfrak{H}$, cf. Definition 3.30) by the following observation. If the set $\mathcal{D}_\iota$ is infinite, the set $\mathcal{D}_{\iota\to o}$ of sets of individuals must be uncountably infinite. On the other hand, any semantics that admits sound and complete calculi must have countable models, because of the compactness theorem that comes with a complete calculus. Leon Henkin generalized the class of admissible domains for functional types. Instead of requiring $\mathcal{D}_{\alpha\to\beta}$ to be the full set of functions, it is sufficient to require that $\mathcal{D}_{\alpha\to\beta}$ has enough members that any well-formed formula can be evaluated[5]. Note that with this generalized notion of a model, there are less formulae that are valid in all models (intuitively, for any given formulae there are more possibilities for counter-models). In the particular case, the generalization to Henkin models, restricts the set of valid formulae sufficiently, so that all of them can be proven by a Hilbert-style calculus [Hen50].

It is matter of folklore that primitive notion equality (expressed by a primitive equality constant $=\in\Sigma$) is not strictly needed, since it can be expressed by the Leibniz formula. However, the Leibniz formula only really denotes the semantic equality relation, if $\mathcal{D}_{\alpha\to o}$ contains enough properties to discern members of α; in fact, we need that for all $a \in \mathcal{D}_\alpha$, the singleton set $\{a\}$ is in $\mathcal{D}_{\alpha\to o}$ (see the proof of Lemma 3.35).[6] In other words, we are in the somewhat paradoxical situation, that Leibniz Equality (which is commonly used as a substitute for primitive equality) will only denote semantical equality, if we can guarantee that the identity relation is already present in the model (we call this property $\mathfrak{q}$, cf. Definition 3.27). Hence we introduce corresponding semantical structures, namely Henkin models without property Q ($\mathfrak{M}_{\mathfrak{fb}}$), in which property $\mathfrak{q}$ is not necessarily valid and thus Leibniz equality does not necessarily denote the equality relation. An example for a theorem which is valid within the class of Henkin models but not in the class of $\mathfrak{M}_{\mathfrak{fb}}$'s, is given by the axiom of functional extensionality for Leibniz equality $(\forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}(\forall X_\beta.FX \doteq GX) \Rightarrow F \doteq^\beta G)$, see lemma 3.36.

The next generalization of model classes comes from the fact that we want to characterize the deductive power of higher-order theorem provers mentioned above on a semantic level (we will take TPS [ABI+96] as an example). Note that TPS cannot be complete with respect to Henkin models and is even not generally complete for $\mathfrak{M}_{\mathfrak{fb}}$'s, although there is some 'extensionality treatment' build into the proof procedure. The uncompleteness of TPS for Henkin models[7] is due to the fact, that it, fails to refute formulae such as $c\mathbf{A}_o \wedge \neg c(\neg\neg\mathbf{A})$, where c is a constant of type $o \to o$ or $c\mathbf{A}_{\alpha\to o} \wedge c\mathbf{B}_{\alpha\to o} \Rightarrow c(\lambda X_\alpha.\mathbf{A}X \wedge \mathbf{B}X)$, where c is a constant of type $(\alpha \to o) \to o$. The problem

[5]In other words: the functional universes are rich enough to satisfy the comprehension axioms.

[6]On a similar note, Peter Andrews remarked in [And72a] that if the set $\mathcal{D}_{\alpha\to\alpha\to o}$ is so sparse, that semantic identity relation is not present, then it is possible to construct a Henkin model, where Leibniz equality is non-extensional.

[7]In case the extensionality axioms are not available in the search space. Note that one can add extensionality axioms to the calculus in order to achieve, at least in theory, Henkin completeness. But this heavily increases the search space and thus is not feasible in practice.

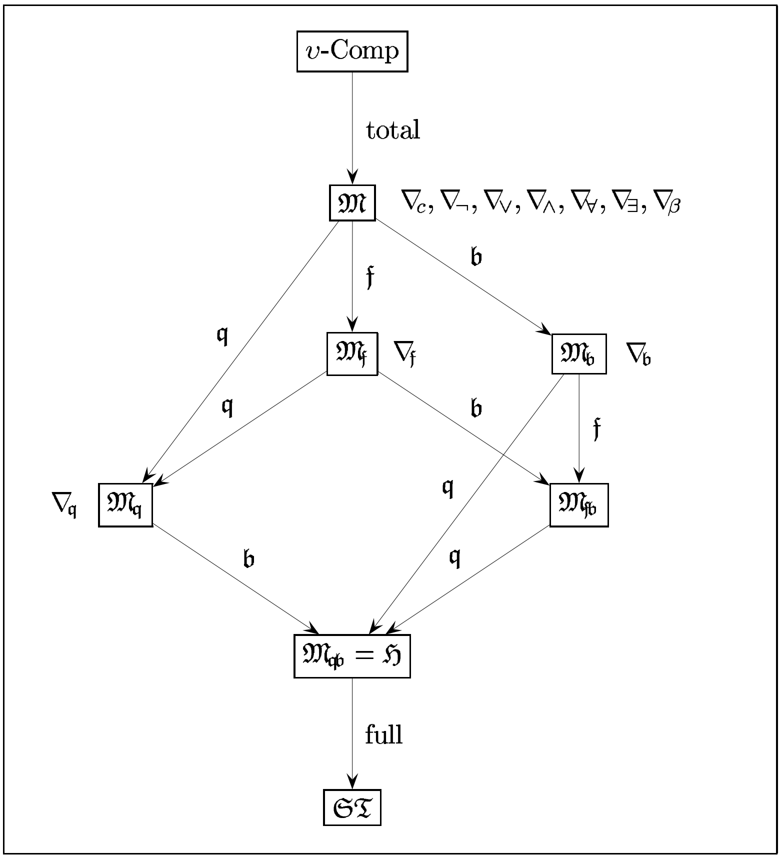

Figure 1: The landscape of Higher-Order Semantics

in the former example is that the higher-order unification algorithm employed by TPS cannot determine that $\mathbf{A}$ and $\neg\neg\mathbf{A}$ denote identical semantic objects (by the extensionality principle on truth values), and thus returns failure instead of success. In the second example in addition to this the principle of functional extensionality is needed in order to prove the theorem.

The lack of completeness of refutation procedures like TPS occurs especially in these situations, where $\mathcal{HOL}$-formulae contain occurrences of propositional formulae dominated by uninterpreted constants or variables or where this problem is mixed with the problem of functional extensionality; in our examples the function constant c dominates the proposition $\mathbf{A}_o$ or the sets $\mathbf{A}_{\alpha\to o}$ and $\mathbf{B}_{\alpha\to o}$. To give a semantical characterization of the deductive power of the TPS procedure, we have to generalize the class of Henkin models further, so that there are counter-models to the examples above. Obviously, this involves weakening the assumption that $\mathcal{D}_o \equiv \{\mathtt{T}, \mathtt{F}\}$ (we call this assumption for Henkin models property $\mathfrak{b}$), since this entails that the values of $\mathbf{A}$ and $\neg\neg\mathbf{A}$ are identical: In **functional Σ-models** ($\mathfrak{M}_\mathfrak{f}, \mathfrak{M}_{\mathfrak{fb}}, \mathfrak{M}_\mathfrak{q}$, cf. Definitions 3.28 and 3.24) we only insist that there is a valuation υ of $\mathcal{D}_o$, i.e. a function $\upsilon\colon \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ that is coordinated with the functions $\mathcal{I}(\neg)$, $\mathcal{I}(\wedge)$, $\mathcal{I}(\Pi^\alpha)$ and (possibly) $\mathcal{I}(=^\alpha)$. Thus we have a notion of validity for Σ: we call a proposition $\mathbf{A}$ valid in $\mathcal{M} := (\mathcal{D}, \mathcal{I}, \upsilon)$ under an assignment φ, iff $\upsilon(\mathcal{I}_\varphi(\mathbf{A})) \equiv \mathtt{T}$. In our first example, there is a Σ-model structure $\mathcal{M} \equiv (\mathcal{D}, \mathcal{I}, \upsilon)$, where $\mathcal{I}_\varphi(\mathbf{A}) \not\equiv \mathcal{I}_\varphi(\neg\neg\mathbf{A})$ and therefore $\mathcal{I}_\varphi(c\mathbf{A}) \not\equiv \mathcal{I}_\varphi(c(\neg\neg\mathbf{A}))$, if we take $\mathcal{I}(c)$ to be the identity function on $\mathcal{D}_\alpha$. In particular, we can have $\upsilon(\mathcal{I}_\varphi(c\mathbf{A})) \not\equiv \upsilon(\mathcal{I}_\varphi(c(\neg\neg\mathbf{A})))$, and therefore $\upsilon(\mathcal{I}_\varphi(c\mathbf{A}_o \wedge \neg c(\neg\neg\mathbf{A}))) \equiv \mathtt{T}$, since υ is a valuation.

Clearly, for functional Σ-models we have the same choices about the role of equality, therefore, we distinguish the classes $\mathfrak{M}_\mathfrak{f}$ and $\mathfrak{M}_\mathfrak{q}$ of functional Σ-models without/with property $\mathfrak{q}$. Furthermore, we have the class $\mathfrak{M}_{\mathfrak{fb}}$ of functional Σ-models with (only) property $\mathfrak{b}$, i.e. where $\mathcal{D}_o \equiv \{\mathtt{T}, \mathtt{F}\}$.

Since functional Σ-models with properties $\mathfrak{b}$ and $\mathfrak{q}$ are defined to be Σ-Henkin models, we can also view $\mathfrak{M}_{\mathfrak{fb}}$ as "Henkin models without property $\mathfrak{q}$".

Finally, we even drop the requirement of functional extensionality for **Σ-models** (cf. Definition 3.24). This is the most general semantical notion that we will discuss in this paper; we only insist that the evaluation function is a homomorphism which respects instantiation. In such models, a function is not uniquely determined by it's behavior on all possible arguments, therefore for the construction of such models we need labelings for functions (e.g. a green and a red version of a function f) that allow to discern them, even though they are functionally equivalent. As done for functional Σ-models, we analyze properties $\mathfrak{q}$ and $\mathfrak{b}$ for non-functional Σ-Models. Whereas $\mathfrak{b}$ indeed may or may not hold for non-functional Σ-Models, it turns out that property $\mathfrak{q}$ implies functionality and hence there are no non-functionalΣ-Models with property $\mathfrak{q}$.

Peter Andrews has pioneered the construction of non-functional models with his v-complexes in [And71]. These are even more general constructions than our Σ-models, since totality of the evaluation function is not assumed. His construction is based on Schütte's semi-valuation method [Sch60], which only needs partial valuations to construct a model for a given Hintikka set.

In this paper, we concentrate on the other aspects of higher-order models and ensure totality of our evaluation functions by a saturation condition (cf. 4.9) in our abstract consistency classes. This does not restrict the applicability of our model existence theorems, since saturation is relatively simple to prove for a given calculus (see [Koh98, BK97]). For all of the notions of models (except naturally for standard models, where such a theorem cannot hold), we present model existence theorems tying the differentiating conditions of the models to suitable conditions in the abstract consistency classes (see section 4.4). We can use the classical construction in all cases: abstract consistent sets are extended to Hintikka sets (see section 4.2), which induce a valuation on a term structure (see Definition 3.14). In some cases, we have to pass to a quotient structure (see Definition 3.12) to ensure that the set of truth values is exactly $\{\mathtt{T}, \mathtt{F}\}$ for property $\mathfrak{b}$.

The simplest way to ensure property $\mathfrak{q}$ is by assuming that the signature contains a primitive logical constant for equality, which is evaluated as semantical identity (we call this property $\mathfrak{e}$). We will study the case in section 4.3. On the one hand, the semantical situation becomes simpler (see figure 2), since $\mathfrak{M}$, $\mathfrak{M}_{\mathfrak{f}}$ and $\mathfrak{M}_{\mathfrak{q}}$ are identified, just as $\mathfrak{M}_{\mathfrak{b}}$, $\mathfrak{M}_{\mathfrak{fb}}$ and $\mathfrak{H}$. On the other hand, the existence of another logical constant induces further conditions in the definition of the abstract consistency classes.

Finally in section 4.5 we refine our methods further by parameterizing them with a type order k and by requiring the function universes with an order greater than k to be full, i.e. to contain all functions. With this modified semantical notions it is possible to restrict the conditions in the abstract consistency classes with respect to the order k. Concretely, the possible instantions of universally quantified formulas can be restricted to terms with an order less or equal to k. With this result it becomes possible to show that the primitive substitution rule in the refutation calculi as discussed in [And71] or [BK97, Koh98] can be restricted with respect to the order of the input problems without loosing completeness.

2.3 Applications

Applications of the results presented in this paper, not only comprise automated theorem proving, where calculus development up to now has been guided by Andrew's "Unifying Principle for Type Theory" [And71]. This model existence theorem has set the completeness standard for higher-order calculi such as [Hue73, ALCMP84], even though it is weaker than the intuitive one given by Henkin Models. The semantical notions in section 3 come from the attempt to achieve completeness with respect to Henkin models for higher-order tableaux [Koh95, Koh98] and higher-order resolution [Koh94a, Ben97, BK97].

A model existence theorem for a logical system $\mathcal{L}$ is a theorem of the form: *If a set of sentences Φ in $\mathcal{L}$ is a member of an abstract consistency class Γ, then there exists a $\mathcal{L}$-model for Φ.* Thus if we want to show the completeness of a particular calculus $\mathcal{C}$, we first prove that the class Γ of sets of sentences Φ that are $\mathcal{C}$-consistent (cannot be refuted in $\mathcal{C}$) is an abstract consistency class, then the model existence theorem tells us that $\mathcal{C}$-consistent sets of sentences are satisfiable in $\mathcal{L}$. Now we assume that a sentence $\mathbf{A}$ is valid in $\mathcal{L}$, so $\neg\mathbf{A}$ does not have a $\mathcal{L}$-model and is therefore $\mathcal{C}$-inconsistent. From this it is easy to verify that $\mathbf{A}$ is a theorem of $\mathcal{C}$. Note that with this argumentation the completeness proof for $\mathcal{C}$ condenses to verifying that Γ is an abstract consistency class, a task that does not refer to $\mathcal{L}$-models. Thus the usefulness of model existence theorems derives from the fact that it replaces the model-theoretic analysis in completeness proofs with the verification of some proof-theoretic conditions (membership in Γ). In this respect a model existence theorem is similar to a Herbrand Theorem, but it is easier to generalize to other logic systems like higher-order logic. The technique was developed for first-order logic by J. Hintikka and R. Smullyan [Hin55, Smu63, Smu68].

Another application of model existence theorems is that they allow for very simple (but non-constructive) proofs of cut-elimination theorems. In [And71] Peter Andrews applies his "Unifying Principle" to cut-elimination in a non-extensional sequent calculus, by proving the calculus complete (relative to $\mathfrak{T}$) both with and without the cut rule and concludes that cut-elimination is valid for this calculus. In the extensional case, where a cut-elimination theorem can be found in [Tak68, Tak87], we can directly model a cut-elimination proof after Andrews' approach, using the model existence theorem for Henkin models.

A related application lies in proof transformation for higher-order logics [Mil83, Pfe87]. Here, proofs found by higher-order automated theorem provers can be transformed into other calculi, such as natural deduction- or sequent calculi that form the basis of tactic-based theorem provers for classical logics like ISABELLE [Pau94] or ΩMEGA [BCF$^+$97]. Dale Miller's original proof transformation system for TPS' [Mil83], uses Andrews' "Unifying principle" and only works for non-extensional calculi like higher-order matings. Frank Pfenning's later extensions (by equality and extensionality) build on various cut-elimination theorems. Again, the methods developed in this paper can shed some light on the situation.

In all these applications, the leverage added by this paper is that we can now extend non-extensional results to extensional cases. However, the generalized model classes have a merit of their own, for instance in higher-order logic programming [NM94], where the denotational semantics of programs can induce non-standard meanings for the classical connectives. For instance, given a SLD-like search strategy as in λ-PROLOG [Mil91], conjunction is not commutative any more. Therefore, various authors have proposed model-theoretic semantics, where property $\mathfrak{b}$ fails. For instance David Wolfram uses Andrew's υ-complexes [Wol94] as a semantics for λ-PROLOG and Gopalan Nadathur uses "labelled structures" for the same purpose in [NM94]. It is plausible to assume that the results of this paper will be useful for further development in this direction.

3 Semantics for Higher Order Logic

In this section we will introduce the semantical constructions and discuss their relationships. We will start out with by defining Σ-structures (and as an intermediate step pre-Σ-structures) as algebraic semantics for the simply typed λ-calculus and then specializing them to various notions of models by requiring a special treatment of propositional formulae.

3.1 Pre-Σ-Structures

Definition 3.1 (Pre-Σ-Structure). A collection $\mathcal{D} := \mathcal{D}_\mathcal{T} := \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$ of sets $\mathcal{D}_\alpha$, indexed by the set $\mathcal{T}$ of types, is called a **typed collection (of sets)**. Let $\mathcal{D}_\mathcal{T}$ and $\mathcal{E}_\mathcal{T}$ be typed collections, then a collection $\mathcal{I} := \{\mathcal{I}^\alpha : \mathcal{D}_\alpha \to \mathcal{E}_\alpha \mid \alpha \in \mathcal{T}\}$ of mappings is called a **typed mapping** $\mathcal{I}: \mathcal{D}_\mathcal{T} \longrightarrow \mathcal{E}_\mathcal{T}$. indexpre-$\Sigma$-structure We call the triple $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ a **pre-Σ-structure**, iff $\mathcal{D} = \mathcal{D}_\mathcal{T}$, is a typed collection of sets and

$$@ := \{@^{\alpha\beta} : \mathcal{D}_{\alpha\to\beta} \times \mathcal{D}_\alpha \longrightarrow \mathcal{D}_\beta \mid \alpha, \beta \in \mathcal{T}\}$$

and $\mathcal{I}: \Sigma \longrightarrow \mathcal{D}$ are typed total functions.

The collection $\mathcal{D}$ is called the **frame of** $\mathcal{A}$, the set $\mathcal{D}_\alpha$ the **universe of type** α, the function $@$ the **application operator**, and the function $\mathcal{I}$ the **interpretation of constants**.

We call a pre-Σ-structure $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ **functional**, iff the following statement holds for all $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$: $\mathsf{f} \equiv \mathsf{g}$, if for all $\mathsf{a} \in \mathcal{D}_\alpha$ $\mathsf{f}@\mathsf{a} \equiv \mathsf{g}@\mathsf{a}$. Note that functionality only poses a restriction on the function universes.

Remark 3.2. The application operator @ in a pre-Σ-structure is an abstract version of function application. It is no restriction to exclusively use a binary application operator, which corresponds to unary function application, since we can define higher-arity application operators from the binary one by setting ("Currying")

$$\mathsf{f}@(\mathsf{a}^1, \ldots, \mathsf{a}^n) := (\ldots(\mathsf{f}@\mathsf{a}^1)\ldots@\mathsf{a}^n)$$

Example 3.3. If we define $\mathbf{A}@\mathbf{B} := (\mathbf{A}\mathbf{B})$ for $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$ and $\mathbf{B} \in \mathit{wff}_\beta(\Sigma)$, then $@\colon \mathit{wff}_{\alpha\to\beta}(\Sigma) \times \mathit{wff}_\alpha(\Sigma) \longrightarrow \mathit{wff}_\beta(\Sigma)$ is a total function. Thus $(\mathit{wff}(\Sigma), @, \mathrm{Id}_\Sigma)$ is a pre-Σ-structure. The intuition behind this example is that we can think of the formula $\mathbf{A} \in \mathit{wff}_{\alpha\to\beta}(\Sigma)$ as a function $\mathbf{A}\colon \mathit{wff}_\alpha(\Sigma) \longrightarrow \mathit{wff}_\beta(\Sigma)$; $\mathbf{B} \mapsto (\mathbf{A}\mathbf{B})$.

Analogously, we can define the pre-Σ-structure $(\mathit{cwff}(\Sigma), @, \mathrm{Id}_\Sigma)$ of closed formulae.

Example 3.4. The following are (trivial) examples for functional pre-Σ-structures:

1. $(\emptyset \times \mathcal{T}, \emptyset, \emptyset)$ is the **empty pre-Σ-structure** and
2. $(\{\mathsf{a}\} \times \mathcal{T}, @^{\mathsf{a}}, \mathcal{I}^{\mathsf{a}})$, where $\mathsf{a}@^{\mathsf{a}}\mathsf{a} \equiv \mathsf{a}$ and $\mathcal{I}^{\mathsf{a}}(c) \equiv \mathsf{a}$ for all constants $c \in \Sigma$ is called the **singleton pre-Σ-structure.**

Definition 3.5 (Σ-Homomorphism). Let $\mathcal{A} := (\mathcal{D}, @^{\mathcal{A}}, \mathcal{I})$ and $\mathcal{B} := (\mathcal{E}, @^{\mathcal{B}}, \mathcal{J})$ be pre-Σ-structures. A **Σ-homomorphism** is a typed function $\kappa\colon \mathcal{D} \longrightarrow \mathcal{E}$ such that

1. $\kappa \circ \mathcal{I} \equiv \mathcal{J}$.
2. For all $\mathsf{f} \in \mathcal{D}_{\alpha\to\beta}$ and $\mathsf{g} \in \mathcal{D}_\alpha$ we have: $\kappa(\mathsf{f})@^{\mathcal{B}}\kappa(\mathsf{g}) \equiv \kappa(\mathsf{f}@^{\mathcal{A}}\mathsf{g})$.

The most important method for constructing Σ-structures with given properties in this paper is well-known for algebraic structures and consists in building a suitable Σ-Congruence and passing to the quotient structure. We will now develop the formal basis for it.

Definition 3.6 (Σ-Congruence). Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ be a pre-Σ-structure, then a typed equivalence relation $\sim$ is called a **Σ-congruence on** $\mathcal{A}$, iff $\mathsf{f} \sim \mathsf{f}' \in \mathcal{D}_{\alpha\to\beta}$ and $\mathsf{g} \sim \mathsf{g}' \in \mathcal{D}_\alpha$ imply $\mathsf{f}@\mathsf{g} \sim \mathsf{f}'@\mathsf{g}'$.

It is called **functional**, iff for all types α, β and all $\mathsf{f}, \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$ the fact that $\mathsf{f}@\mathsf{a} \sim \mathsf{g}@\mathsf{a}$ for all $\mathsf{a} \in \mathcal{D}_\beta$ implies $\mathsf{f} \sim \mathsf{g}$. Note that, since $\sim$ is a congruence, we also have the other direction so we have

$$\mathsf{f}@\mathsf{a} \sim \mathsf{g}@\mathsf{a} \text{ for all } \mathsf{a} \in \mathcal{D}_\beta, \text{ iff } \mathsf{f} \sim \mathsf{g}$$

Lemma 3.7. *The β and $\beta\eta$ equality relations $\overset{*}{\leftrightarrow}_\beta$ and $\overset{*}{\leftrightarrow}_{\beta\eta}$ are congruences on the pre-Σ-structures $\mathit{wff}(\Sigma)$ and $\mathit{cwff}(\Sigma)$ by definition. Moreover, $\beta\eta$-equality is functional $\mathit{wff}(\Sigma)$ and $\mathit{cwff}(\Sigma)$.*

Proof: The congruence properties are a direct consequence of the fact that $\beta\eta$ reduction rules are defined to act on sub-term positions. We will establish functionality of $\overset{*}{\leftrightarrow}_{\beta\eta}$ on $\mathit{wff}(\Sigma)$ first and then use this to obtain the assertion for closed formulae.

Let $\mathbf{A}_{\gamma\to\alpha}\mathbf{C}_\gamma \overset{*}{\leftrightarrow}_{\beta\eta} \mathbf{B}_{\gamma\to\alpha}\mathbf{C}$ for all $\mathbf{C}$, then in particular, for any variable $X \in \mathcal{V}_\gamma$ that is not free in $\mathbf{A}$ or $\mathbf{B}$, we have $\mathbf{A}X \overset{*}{\leftrightarrow}_{\beta\eta} \mathbf{B}X$ and $\lambda X \mathbf{A}X \overset{*}{\leftrightarrow}_{\beta\eta} \lambda X \mathbf{B}X$. By definition we have $\mathbf{A} \overset{*}{\leftrightarrow}_\eta \lambda X_\alpha . \mathbf{A}X \overset{*}{\leftrightarrow}_{\beta\eta} \lambda X_\alpha \mathbf{B}X \overset{*}{\leftrightarrow}_\eta \mathbf{B}$.

To show functionality of $\beta\eta$ on closed formulae, let $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_{\alpha\to\beta}(\Sigma)$, such that $\mathbf{A} \overset{*}{\not\leftrightarrow}_{\beta\eta} \mathbf{B}$. Since $\beta\eta$ is functional on $\mathit{wff}(\Sigma)$, there must be a formula $\mathbf{C}$ with $\mathbf{A}\mathbf{C} \overset{*}{\not\leftrightarrow}_{\beta\eta} \mathbf{B}\mathbf{C}$. Now let $\mathbf{C}'$ be a ground instance of $\mathbf{C}$, i.e. $\mathbf{C}' = \sigma(\mathbf{C})$, where σ is a closed substitution[8], then we have $\mathbf{A}\mathbf{C}' \overset{*}{\not\leftrightarrow}_{\beta\eta} \mathbf{B}\mathbf{C}'$. Thus we have shown that $\mathbf{A} \overset{*}{\not\leftrightarrow}_{\beta\eta} \mathbf{B}$ entails $\mathbf{A}\mathbf{C}' \overset{*}{\not\leftrightarrow}_{\beta\eta} \mathbf{B}\mathbf{C}'$, which gives us the assertion. □

[8]This has to exist, since we have assumed all types to be non-empty.

Definition 3.8 (Quotient Pre-Σ-Structure). Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ be a pre-Σ-structure, $\mathcal{D}_\alpha^\sim := \{[\![\mathsf{f}]\!] \mid \mathsf{f} \in \mathcal{D}_\alpha\}$, and $\mathcal{I}^\sim(c_\alpha) := [\![\mathcal{I}(c_\alpha)]\!]$ for all constants $c_\alpha \in \Sigma_\alpha$. Furthermore let $@^\sim$ be defined by $[\![\mathsf{f}]\!]@^\sim[\![\mathsf{a}]\!] := [\![\mathsf{f}@\mathsf{a}]\!]$. To see that this definition only depends only on equivalence classes of $\sim$, consider $\mathsf{f}' \in [\![\mathsf{f}]\!]$ and $\mathsf{g}' \in [\![\mathsf{g}]\!]$, then $[\![\mathsf{f}@\mathsf{g}]\!] \equiv [\![\mathsf{f}'@\mathsf{g}]\!] \equiv [\![\mathsf{f}'@\mathsf{g}']\!] \equiv [\![\mathsf{f}@\mathsf{g}']\!]$. So $@^\sim$ is well-defined and total, thus $\mathcal{A}/_\sim := (\mathcal{D}^\sim, @^\sim, \mathcal{I}^\sim)$ is also a pre-Σ-structure. We call $\mathcal{A}/_\sim$ the **quotient structure of $\mathcal{A}$ for the relation $\sim$** and the typed function $\pi_\sim : \mathcal{A} \longrightarrow \mathcal{A}/_\sim; \mathsf{f} \mapsto [\![\mathsf{f}]\!]$ its **canonical projection.**

This definition is justified by the following theorem.

Theorem 3.9. *Let $\mathcal{A}$ be a pre-Σ-structure and let $\sim$ be an Σ-congruence on $\mathcal{A}$, then the canonical projection $\pi_\sim$ is a surjective Σ-homomorphism. Furthermore, $\mathcal{A}/_\sim$ is functional, iff $\sim$ is functional.*

Proof: Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ be a pre-Σ-structure. To convince ourselves that $\pi_\sim$ is indeed a surjective Σ-homomorphism, we note that by definition $\pi_\sim$ is surjective and $\mathcal{I}^\sim \equiv \pi_\sim \circ \mathcal{I}$. Now let $\mathsf{f} \in \mathcal{D}_{\beta\to\alpha}$, and $\mathsf{g} \in \mathbf{Dom}(\mathsf{f}) \subseteq \mathcal{D}_\beta$, then $\mathsf{g}' \in [\![\mathsf{g}]\!]$ for all $\mathsf{g}' \in \mathbf{Dom}(\mathsf{f})$ and therefore $[\![\mathsf{g}]\!] \equiv \pi_\sim(\mathsf{g}) \in \mathbf{Dom}([\![\mathsf{f}]\!]) \equiv \mathbf{Dom}(\pi_\sim(\mathsf{f}))$ and $\pi_\sim(\mathsf{f})@^\sim\pi_\sim(\mathsf{g}) \equiv [\![\mathsf{f}]\!]@^\sim[\![\mathsf{g}]\!] \equiv [\![\mathsf{f}@\mathsf{g}]\!] \equiv \pi_\sim(\mathsf{f}@\mathsf{g})$.

The quotient construction trivializes $\sim$ to (meta-)equality, so functionality of $\sim$ is equivalent to functionality of $\mathcal{A}$. Formally we have $[\![\mathsf{f}]\!] \equiv [\![\mathsf{g}]\!]$, iff $\mathsf{f} \sim \mathsf{g}$, iff $f@\mathsf{a} \sim \mathsf{g}@\mathsf{a}$, iff $[\![\mathsf{f}@\mathsf{a}]\!] \equiv [\![\mathsf{g}@\mathsf{a}]\!]$, iff $[\![\mathsf{f}]\!]@^\sim[\![\mathsf{a}]\!] \equiv [\![\mathsf{g}]\!]@^\sim[\![\mathsf{a}]\!]$ for all $\mathsf{a} \in \mathcal{D}_\alpha$ and thus for all $[\![\mathsf{a}]\!] \in \mathcal{D}_\alpha^\sim$. $\square$

3.2 Σ-Structures

Σ-structures are pre-Σ-structures with a notion of evaluation for *wff*(Σ).

Definition 3.10 (Σ-Structure). Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ be a pre-Σ-structure. A typed function $\varphi : \mathcal{V} \longrightarrow \mathcal{D}$ is called an **assignment into $\mathcal{A}$**. We call a total typed mapping[9] $\mathcal{E} : \mathcal{F}(\mathcal{V}; \mathcal{D}) \times wff(\Sigma) \longrightarrow \mathcal{D}$ an **evaluation function** for $\mathcal{A}$, iff for any assignment φ into $\mathcal{A}$, we have

1. $\mathcal{E}_\varphi|_\Sigma \equiv \mathcal{I}$ and $\mathcal{E}_\varphi|_\mathcal{V} \equiv \varphi$
2. $\mathcal{E}_\varphi$ is a Σ-homomorphism
3. $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\psi(\mathbf{A})$, whenever φ and ψ coincide on $\mathbf{Free}(\mathbf{A})$
4. $\mathcal{E}_\varphi([\mathbf{B}/X]\mathbf{A}) \equiv \mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\mathbf{B})/X]}(\mathbf{A})$

We call $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ **Σ-structure**, iff $(\mathcal{D}, @, \mathcal{I})$ is a pre-Σ-structure and $\mathcal{E}$ is an evaluation function for $\mathcal{A}$. We call $\mathcal{E}_\varphi(\mathbf{A}_\alpha) \in \mathcal{D}_\alpha$ **denotation of $\mathbf{A}_\alpha$ in $\mathcal{A}$ for φ.**

If $\mathbf{A}$ is a closed formula, then $\mathcal{E}_\varphi(\mathbf{A})$ is independent of φ, since $\mathbf{Free}(\mathbf{A}) = \emptyset$. In these cases we sometimes drop the reference from $\mathcal{E}_\varphi(\mathbf{A})$ and simply write $\mathcal{E}(\mathbf{A})$.

Example 3.11. The pre-Σ-structure $\mathcal{T} \times \emptyset$ cannot be a Σ-structure, since we must have $\mathcal{E}_\varphi(\lambda X_\alpha.X) \in \mathcal{D}_{\alpha\to\alpha}$. In contrast to this, the singleton pre-Σ-structure is a Σ-structure if we take $\mathcal{E}(\mathbf{A}) \equiv a$, where a is the (unique) member of $\mathcal{D}_\alpha$.

For a detailed discussion on the closure conditions needed for the function universes to be rich enough, we refer the reader to [And72a, And73].

Note that the pre-Σ-structure *wff*(Σ) from 3.3 cannot be made into a Σ-structure by providing an evaluation function, since there is no formula $\mathbf{C} \equiv \mathcal{I}_\varphi(\lambda X_\alpha.\mathbf{B}) \in wff_{\alpha\to\beta}(\Sigma)$ such that $\mathbf{C}@\mathbf{A} \equiv \mathbf{CA} \equiv \mathcal{I}_{\varphi,[\mathbf{A}/X]}(\mathbf{B})$. In particular, the "obvious" choice $\lambda X_\alpha.\mathbf{B}$ for $\mathbf{C}$ does not work, since $(\lambda X_\alpha.\mathbf{B})\mathbf{A} \not\equiv \mathcal{I}_{\varphi,[\mathbf{A}/X]}(\mathbf{B})$. In fact, if *wff*($\Sigma$) were a Σ-structure, β-equality would have to be valid in *wff*(Σ) (cf. 3.17), which it clearly is not.

Definition 3.12 (Quotient Σ-Structure). Let $\mathcal{A} = (\mathcal{D}, @, \mathcal{E})$ be a Σ-structure, $\sim$ a Σ-congruence on $\mathcal{A}$ and let $\mathcal{A}/_\sim = (\mathcal{D}^\sim, @^\sim, \mathcal{I}^\sim)$ be the quotient pre-Σ-structure of $\mathcal{A}$, where $\mathcal{I} = \mathcal{E}|_\Sigma$.

For any assignment ψ $\mathcal{A}/_\sim$, there exists an assignment φ into $\mathcal{A}$ such that $\psi \equiv \pi_\sim \circ \varphi$, since $\pi_\sim$ is a surjective Σ-homomorphism. So we can define $\mathcal{E}_\varphi^\sim$ as $\pi_\sim \circ \mathcal{E}_\psi$, and call $\mathcal{A}/_\sim := (\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim)$ the **quotient Σ-structure** of $\mathcal{A}$ modulo $\sim$.

Theorem 3.13. *Let $\mathcal{A}$ be a Σ-structure and $\sim$ a Σ-congruence on $\mathcal{A}$, then $\mathcal{A}/_\sim$ is a Σ-structure.*

[9]We write $\mathcal{F}(\mathcal{V}; \mathcal{D})$ for the set of functions $f : \mathcal{V} \to \mathcal{D}$

Proof: We prove that $\mathcal{E}^{\sim}$ is a legal value function by verifying the conditions in 3.10: Let φ and ψ be assignments, such that $\psi \equiv \pi_{\sim} \circ \varphi$, then

1. $\mathcal{E}^{\sim}_{\varphi}\big|_{\Sigma} \equiv (\pi_{\sim} \circ \mathcal{E}_{\psi})\big|_{\Sigma} \equiv \pi_{\sim} \circ \mathcal{E}_{\psi}\big|_{\Sigma} \equiv \pi_{\sim} \circ \mathcal{I} \equiv \mathcal{I}^{\sim}$ and $\mathcal{E}^{\sim}_{\varphi}\big|_{\mathcal{V}} \equiv (\pi_{\sim} \circ \mathcal{E}_{\psi})\big|_{\mathcal{V}} \equiv \pi_{\sim} \circ \mathcal{E}_{\psi}\big|_{\mathcal{V}} \equiv \pi_{\sim} \circ \psi \equiv \varphi$
2. $\mathcal{E}^{\sim}_{\varphi} \equiv \pi_{\sim} \circ \mathcal{E}_{\psi}$ is a Σ-homomorphism, since $\pi_{\sim}$ and $\mathcal{E}_{\psi}$ are.
3. $\mathcal{E}^{\sim}_{\varphi}(\mathbf{A}) \equiv [\![\mathcal{E}_{\psi}(\mathbf{A})]\!] \equiv [\![\mathcal{E}_{\psi'}(\mathbf{A})]\!] \equiv \mathcal{E}^{\sim}_{\varphi'}(\mathbf{A})$, iff φ and φ' coincide on $\mathbf{Free}(\mathbf{A})$, since this entails that ψ and ψ' do too.
4. $\mathcal{E}^{\sim}_{\varphi}([\mathbf{B}/X]\mathbf{A}) \equiv [\![\mathcal{E}_{\psi}([\mathbf{B}/X]\mathbf{A})]\!] \equiv [\![\mathcal{E}_{\psi,[\mathcal{E}_{\psi}(\mathbf{B})/X]}(\mathbf{A})]\!] \equiv \mathcal{E}^{\sim}_{\varphi,[\mathcal{E}^{\sim}_{\varphi}(\mathbf{B})/X]}(\mathbf{A})$, since $[\![\mathcal{E}_{\psi}(\mathbf{B})]\!] \equiv \mathcal{E}^{\sim}_{\varphi}(\mathbf{B})$ and therefore $\pi_{\sim} \circ \psi, [\mathcal{E}_{\psi}(\mathbf{B})/X] \equiv \varphi, [\mathcal{E}^{\sim}_{\varphi}(\mathbf{B})/X]$ □

Definition 3.14 (Term Structures for Σ). Let $\mathit{cwff}(\Sigma)_{\downarrow_\beta}$ be the collection of well-formed formulae in β-normal form and $\mathbf{A}@^{\beta}\mathbf{B}$ be the β-normal form of $\mathbf{AB}$. For the definition of an evaluation function let φ be an assignment into $\mathit{cwff}(\Sigma)_{\downarrow_\beta}$. Note that $\sigma := \varphi|_{\mathbf{Free}(\mathbf{A})}$ is a substitution, since $\mathbf{Free}(\mathbf{A})$ is finite. Thus we can choose $\mathcal{E}^{\beta}_{\varphi}(\mathbf{A}) := \sigma(\mathbf{A})_{\downarrow_\beta}$, where $\mathbf{A}_{\downarrow_\beta}$ is the β-normal form of $\mathbf{A}$. We call $\mathcal{TS}(\Sigma)^{\beta} := (\mathit{cwff}(\Sigma)_{\downarrow_\beta}, @^{\beta}, \mathcal{E}^{\beta})$ the **β-term structure for Σ**.

Analogously, we can define $\mathcal{TS}(\Sigma)^{\beta\eta} := (\mathit{cwff}(\Sigma)_{\downarrow_{\beta\eta}}, @^{\beta\eta}, \mathcal{E}^{\beta\eta})$ the **$\beta\eta$-term structure for Σ**.

The name "term structure" in the previous definition is justified by the following lemma.

Lemma 3.15. *$\mathcal{TS}(\Sigma)^{\beta}$ is Σ-structure and $\mathcal{TS}(\Sigma)^{\beta\eta}$ is a functional Σ-structure.*

Proof: Note that constants are β-normal forms, therefore $\mathcal{TS}(\Sigma)^{\beta}$ is the quotient structure of $\mathit{cwff}(\Sigma)$ for the congruence $\overset{*}{\leftrightarrow}_{\beta}$. As we have remarked in 3.11, $\mathit{wff}(\Sigma)$ is not a Σ-structure, so we cannot use 3.13, but have to convince ourselves directly that $\mathcal{TS}(\Sigma)^{\beta}$ is a Σ-structure by verifying the conditions of 3.10. The first three are direct consequences of the definition of $\mathcal{E}^{\beta}$ as substitution application.

1. $\mathcal{E}^{\beta}_{\varphi}\big|_{\Sigma} \equiv \mathcal{I}^{\beta} = \mathrm{Id}_{\Sigma}$ and $\mathcal{E}^{\beta}_{\varphi}\big|_{\mathcal{V}} \equiv \varphi$
2. $\mathcal{E}^{\beta}_{\varphi}$ is a Σ-homomorphism
3. $\mathcal{E}^{\beta}_{\varphi}(\mathbf{A}) \equiv \sigma(\mathbf{A}) \equiv \sigma'(\mathbf{A}) \equiv \mathcal{E}_{\varphi'}(\mathbf{A})$, iff φ and φ' coincide on $\mathbf{Free}(\mathbf{A})$
4. $\mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A}) \equiv \sigma([\mathbf{B}/X]\mathbf{A}) \equiv [\sigma(\mathbf{B})/X](\sigma'(\mathbf{A})) \equiv \sigma, [\sigma(\mathbf{B})/X]\mathbf{A} \equiv \mathcal{E}^{\beta}_{\varphi,[\mathcal{E}^{\beta}_{\varphi}(\mathbf{B})/X]}(\mathbf{A})$, where σ' is $\sigma, [X/X]$.

Since $\overset{*}{\leftrightarrow}_{\beta\eta}$ is a sub-relation of $\overset{*}{\leftrightarrow}_{\beta}$, an analogous argumentation shows that $\mathcal{TS}(\Sigma)^{\beta\eta}$ is a Σ-structure. Furthermore, $\overset{*}{\leftrightarrow}_{\beta\eta}$ is a functional Σ-congruence on $\mathit{wff}(\Sigma)$ (cf. 3.7), so we know by 3.9 that $\mathcal{TS}(\Sigma)^{\beta\eta}$ is functional. □

Remark 3.16. Note that $\mathcal{TS}(\Sigma)^{\beta}$ is not a functional Σ-structure since e.g. $(\lambda X_{\gamma}.Y_{\gamma\to\delta}X)@^{\beta}\mathbf{C}_{\gamma} \equiv \mathbf{Y}@^{\beta}\mathbf{C}$ for all $\mathbf{C}$ in $\mathcal{TS}_{\gamma}(\Sigma)^{\beta}$ but $\lambda X.YX \not\equiv Y$.

In a general Σ-structure $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ constants are given a meaning by the interpretation function $\mathcal{I}\colon \Sigma \to \mathcal{D}$, and variables get their meaning by assignments $\varphi\colon \mathcal{V} \to \mathcal{D}$. Furthermore, the evaluation function has to respect instantiation like in first-order logic. This is enough to ensure soundness of β-equality. We do not have to show soundness of α-equality, since this is trivial as we have assumed alphabetic variants to be identical.

Lemma 3.17 (Soundness of β-equality). *Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ be a Σ-structure and φ an assignment into $\mathcal{A}$, then $\mathcal{E}_{\varphi}((\lambda X.\mathbf{A})\mathbf{B}) \equiv \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A})$ provided that X not bound in $\mathbf{A}$.*

Proof: By the definition of Σ-structures, we have $\mathcal{E}_{\varphi}((\lambda X.\mathbf{A})\mathbf{B}) \equiv \mathcal{E}_{\varphi}(\lambda X.\mathbf{A})@\mathcal{E}_{\varphi}(\mathbf{B}) \equiv \mathcal{E}_{\varphi,[\mathcal{E}_{\varphi}(\mathbf{B})/X]}(\mathbf{A}) \equiv \mathcal{E}_{\varphi}([\mathbf{B}/X]\mathbf{A})$ □

3.3 Functional Σ-structures

For functional Σ-structures, there is another way to define evaluation: Since well-formed formulae are inductively built up from constants and variables we can extend φ and $\mathcal{I}$ to a Σ-homomorphism on well-formed formulae.

Definition 3.18 (Homomorphic Extension). Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ be a functional pre-Σ-structure and let φ be an assignment into $\mathcal{A}$. Then the **homomorphic extension $\mathcal{I}_\varphi$ of φ to** *wff*(Σ) is inductively defined to be a typed partial function $\mathcal{I}_\varphi\colon wff(\Sigma) \longrightarrow \mathcal{D}$ such that

1. $\mathcal{I}_\varphi(X) \equiv \varphi(X)$, if X is a variable,
2. $\mathcal{I}_\varphi(c) \equiv \mathcal{I}(c)$, if c is a constant,
3. $\mathcal{I}_\varphi(\mathbf{AB}) \equiv \mathcal{I}_\varphi(\mathbf{A})@\mathcal{I}_\varphi(\mathbf{B})$,
4. $\mathcal{I}_\varphi(\lambda X_\alpha.\mathbf{B}_\beta)$ is the function in $\mathcal{D}_{\alpha\to\beta}$ such that $\mathcal{I}_\varphi(\lambda X_\alpha.\mathbf{B})@z := \mathcal{I}_{\varphi,[z/X]}(\mathbf{B})$. Note that this function is unique, since we have assumed $\mathcal{A}$ to be functional.

Note that we have to assume that the universes of functions $\mathcal{D}_{\alpha\to\beta}$ are rich enough to contain a value for all $\mathbf{A}_{\alpha\to\beta} \in wff_{\alpha\to\beta}(\Sigma)$ for this construction to yield a total function.

Lemma 3.19. *Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{I})$ be a functional pre-Σ-structure, then $\mathcal{E}\colon \varphi \mapsto \mathcal{I}_\varphi$ is an evaluation function for $\mathcal{A}$.*

Proof: To prove the assertion, we have to show the conditions of 3.10. The first one is trivially met by construction, the second is a direct consequence of the fact that $\mathcal{I}_\varphi \circ \mathrm{Id}_\Sigma \equiv \mathcal{I} \circ \mathrm{Id}_\Sigma \equiv \mathcal{I}$ on Σ.

For the third condition, we prove that the value of a function only depends on its free variables by induction on the structure of $\mathbf{A}$. The only interesting case is the one, where $\mathbf{A}$ is an abstraction, since the assertion is trivial for constants and variables, and a simple consequence of the inductive hypothesis for applications. So let $\mathbf{A} \doteq (\lambda X.\mathbf{B})$, then $\mathcal{I}_\varphi(\mathbf{A})@\mathsf{a} \equiv \mathcal{I}_{\varphi,[\mathsf{a}/X]}(\mathbf{B}) \equiv \mathcal{I}_{\psi,[\mathsf{a}/X]}(\mathbf{B}) \equiv \mathcal{I}_\psi(\mathbf{A})@\mathsf{a}$ by inductive hypothesis, since $\varphi, [\mathsf{a}/X]$ and $\psi, [\mathsf{a}/X]$ coincide on the free variables of $\mathbf{B}$. Thus we obtain the assertion from the definition of $\mathcal{I}_\varphi$.

Finally, we prove the fourth condition by induction on the structure of $\mathbf{A}$. If $\mathbf{A}$ is a constant or variable, then the assertion is trivial. The case where $\mathbf{A}$ is the application $\mathbf{CD}$ is entailed by the fact, that substitution and homomorphic extension are defined inductively on the structure of applications: We have

$$\begin{aligned}
\mathcal{I}_\varphi([\mathbf{B}/X]\mathbf{CD}) &\equiv \mathcal{I}_\varphi([\mathbf{B}/X]\mathbf{C})@\mathcal{I}_\varphi([\mathbf{B}/X]\mathbf{D}) \\
&\equiv \mathcal{I}_{\varphi,[\mathcal{I}_\varphi(\mathbf{B})/X]}(\mathbf{C})@\mathcal{I}_{\varphi,[\mathcal{I}_\varphi(\mathbf{B})/X]}(\mathbf{D}) \\
&\equiv \mathcal{I}_{\varphi,[\mathcal{I}_\varphi(\mathbf{B})/X]}(\mathbf{CD})
\end{aligned}$$

If $\mathbf{A} \equiv (\lambda Y.\mathbf{D})$ and $\psi \equiv \varphi, [a/Y]$, then

$$\mathcal{I}_\varphi([\mathbf{B}/X]\mathbf{A})@a \equiv \mathcal{I}_\varphi(\lambda Y.[\mathbf{B}/X]\mathbf{D})@a \equiv \mathcal{I}_\psi([\mathbf{B}/X]\mathbf{D}) \equiv \mathcal{I}_{\psi,[\mathcal{I}_\psi(\mathbf{B})/X]}(\mathbf{D})$$

by inductive hypothesis. Note that ψ and φ coincide on the free variables of $\mathbf{A}$, therefore by the third condition, which we have proven above, we have $\mathcal{I}_{\psi,[\mathcal{I}_\varphi(\mathbf{B})/X]}(\mathbf{D}) \equiv \mathcal{I}_{\varphi,[\mathcal{I}_\varphi(\mathbf{B})/X]}(\lambda Y.\mathbf{D})@a$, which implies the assertion, since $\mathcal{A}$ is functional. $\square$

In fact, for functional Σ-structures, the two notions of evaluation coincide, as we shall see in the next lemma.

Lemma 3.20 (Evaluation in functional Σ-Structures). *If $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ is a functional Σ-structure, then $\mathcal{E}_\varphi \equiv \mathcal{I}_\varphi$ for any assignment φ into $\mathcal{A}$.*

Proof: Let $\mathbf{A} \in wff(\Sigma)$, we prove the assertion by induction over the size of $\mathbf{A}$. The assertion is trivial, if $\mathbf{A}$ is a constant or variable and a simple consequence of the inductive hypothesis, if $\mathbf{A}$ is an application. So let $\mathbf{A} := (\lambda X.\mathbf{B})$, furthermore let Y be a variable not in $\mathbf{Free}(\mathbf{A})$ and $\psi := \varphi, [\mathsf{a}/Y]$. Then

$$\mathcal{E}_\varphi(\mathbf{A})@\mathsf{a} \equiv \mathcal{E}_\psi(\mathbf{A})@\mathsf{a} \equiv \mathcal{E}_\varphi(\mathbf{A})@\mathcal{E}_\psi(Y) \equiv \mathcal{E}_\psi(\mathbf{A}Y) \equiv \mathcal{E}_\psi([Y/X]\mathbf{B})$$

since β-equality is sound in Σ-structures. Now $[Y/X]\mathbf{B}$ is smaller than $\mathbf{A}$, so we can use the inductive hypothesis to obtain

$$\mathcal{E}_\varphi(\mathbf{A})@a \equiv \mathcal{I}_\psi([Y/X]\mathbf{B}) \equiv \mathcal{I}_\psi(\mathbf{A}Y) \equiv \mathcal{I}_\varphi(\mathbf{A})@\mathcal{I}_\psi(Y) \equiv \mathcal{I}_\varphi(\mathbf{A})@\mathsf{a}$$

which entails the assertion since $\mathcal{A}$ is functional. $\square$

Lemma 3.21. *Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ be a functional Σ-structure and X be a variable that is not free in $\mathbf{A}$, then $\mathcal{E}_\varphi(\lambda X.\mathbf{A}X) \equiv \mathcal{E}_\varphi(\mathbf{A})$ for all assignments φ into $\mathcal{A}$.*

Proof: With 3.10.3 and the fact that X is not free in $\mathbf{A}$ we have

$$\mathcal{E}_\varphi(\lambda X.\mathbf{A}X)@\mathsf{a} \equiv \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{A})@\mathcal{E}_{\varphi,[\mathsf{a}/X]}(X) \equiv \mathcal{E}_\varphi(\mathbf{A})@\mathsf{a}$$

which implies the assertion $\mathcal{E}_\varphi(\lambda X.\mathbf{A}X) \equiv \mathcal{E}_\varphi(\mathbf{A})$, as $\mathcal{A}$ is functional. $\square$

We now specialize the notion of Σ-structures to the standard general model semantics for $\Lambda^{\rightarrow}$.

Definition 3.22 (Σ-Algebra). A **pre-Σ-algebra** $\mathcal{A} := (\mathcal{D}, \mathcal{I})$ is a pre-Σ-structure $(\mathcal{D}, @, \mathcal{I})$ such that $\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{F}(\mathcal{D}_\alpha; \mathcal{D}_\beta)$ and $\mathsf{f}@\mathsf{a} \equiv \mathsf{f}(\mathsf{a})$. A pre-$\Sigma$-algebra is called **full**, iff $\mathcal{D}_{\alpha\to\beta} \equiv \mathcal{F}(\mathcal{D}_\alpha; \mathcal{D}_\beta)$. We call a pre-$\Sigma$-algebra an **$\Sigma$-algebra**, iff it is a Σ-structure.

Remark 3.23. Note that pre-Σ-algebras are functional, since they are defined as structures of mathematical functions. On the other hand, for any functional Σ-structure $\mathcal{A}$, we can define an isomorphic Σ-algebra $\mathcal{A}'$

Proof: For a functional Σ-structure $\mathcal{A} = (\mathcal{D}, @, \mathcal{I})$ we define a Σ-algebra $\mathcal{A}' = (\mathcal{D}', \mathcal{I}')$ and a bijective Σ-homomorphism $\kappa\colon \mathcal{A} \longrightarrow \mathcal{A}'$ by an induction on the type:

- $\mathcal{D}'_\alpha := \mathcal{D}_\alpha$ for all $\alpha \in \mathcal{BT}$ and $\kappa = \mathrm{Id}_\mathcal{D}$; obviously κ is bijective.
- $\mathcal{D}'_{\alpha\to\beta} := \kappa(\mathcal{D}_{\alpha\to\beta})$ and $\kappa(\mathsf{f}) = \kappa \circ (@\mathsf{f}) \circ \kappa^{-1}$ for $\mathsf{f} \in \mathcal{D}_{\alpha\to\beta}$. Note that with this construction κ is a homomorphism, since

  $$\kappa(\mathsf{f})(\kappa(\mathsf{a})) = \kappa(\mathsf{f}@(\kappa^{-1}(\kappa(\mathsf{a})))) = \kappa(\mathsf{f}@\mathsf{a})$$

  κ is surjective by construction and injective, since $\mathcal{A}$ is functional: If $\mathsf{f} \neq \mathsf{g} \in \mathcal{D}_{\alpha\to\beta}$, then there is an $\mathsf{a} \in \mathcal{D}_\alpha$, such that $\mathsf{f}(\mathsf{a}) \neq \mathsf{g}(\mathsf{a})$, in particular, we have

  $$\kappa(\mathsf{f}(\mathsf{a})) = \kappa(\mathsf{f})@\kappa(\mathsf{a}) \neq \kappa(\mathsf{f})@\kappa(\mathsf{a}) = \kappa(\mathsf{g}(\mathsf{a}))$$

  since κ is injective on $\mathcal{D}_\beta$. Thus and therefore $\kappa(\mathsf{f}) \neq \kappa(\mathsf{g})$, since $\kappa(\mathsf{f}), \kappa(\mathsf{g}) \in \mathcal{F}(\mathcal{D}_\alpha; \mathcal{D}_\beta)$

Now, we only have to choose $\mathcal{I}' := \kappa \circ \mathcal{I}$ to complete the construction of $\mathcal{A}'$. $\square$

As a consequence, we can always consider functional Σ-structures as Σ-algebras.

3.4 Σ-Models

Up to now, the semantical notions introduced were totally independent of the set of base types assumed. Now, we specialize these to obtain a notion of models by requiring specialized behavior on the type o of truth values. For this we use the notion of a Σ-valuation, which intuitively gives a truth-value interpretation to the domain $\mathcal{D}_o$ of a Σ-structure, which is consistent with the intuitive interpretations of the logical constants. Since models are semantic entities that are constructed to make statements about truth and falsity of formulae, the requirement that there exists a Σ-valuation is perhaps the most general condition under which one wants to speak of a model. Thus we will define our most general notion of semantics as Σ-structures that have Σ-valuations.

Definition 3.24 (Σ-Model). Let $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ be a Σ-structure, then a surjective total function $\upsilon\colon \mathcal{D}_o \longrightarrow \{\mathtt{T}, \mathtt{F}\}$ such that

1. $\upsilon(\mathcal{E}(\neg)@\mathsf{a}) \equiv \mathtt{T}$, iff $\upsilon(a) \equiv \mathtt{F}$,
2. $\upsilon(\mathcal{E}(\vee)@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$, iff $\upsilon(\mathsf{a}) \equiv \mathtt{T}$ or $\upsilon(\mathsf{b}) \equiv \mathtt{T}$,

3. $\upsilon(\mathcal{E}(\Pi^\alpha)@\mathsf{f}) \equiv \mathtt{T}$, iff $\upsilon(\mathsf{f}@\mathsf{a}) \equiv \mathtt{T}$ for each $\mathsf{a} \in \mathcal{D}_\alpha$

is called a **Σ-valuation for** $\mathcal{A}$ and $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is called a **Σ-model ($\mathfrak{M}$).**

We say that an assignment φ **satisfies** a formula $\mathbf{A} \in wff_o(\Sigma)$ in $\mathcal{M}$ ($\mathcal{M} \models_\varphi \mathbf{A}$), iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathtt{T}$ and that $\mathbf{A}$ is **valid** in $\mathcal{M}$, iff $\mathcal{M} \models_\varphi \mathbf{A}$ for all assignments φ. Finally, we say that $\mathcal{M}$ is a **Σ-model** for a set $H \subseteq wff_o(\Sigma)$ ($\mathcal{M} \models H$) iff $\mathcal{M}$ satisfies all $\mathbf{A} \in H$.

Lemma 3.25 (Truth and Falsity in Σ-models). *Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ an assignment. Furthermore let $\mathbf{T}_o := \mathbf{A}_o \vee \neg(\mathbf{A}_o)$ for some $\mathbf{A}_o \in wff_o$ and let $\mathbf{F}_o := \neg\mathbf{T}_o$. Then $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) \equiv \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) \equiv \mathtt{F}$.*

Proof: We have $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) \equiv \mathtt{T}$ iff $\upsilon(\mathcal{E}_\varphi(\mathbf{A}_o \vee \neg(\mathbf{A}_o))) \equiv \mathtt{T}$. Evaluation shows that this statement is equivalent to $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathtt{T}$ or $\upsilon(\mathcal{E}_\varphi(\mathbf{A})) \equiv \mathtt{F}$, which is valid since $\varphi : \mathcal{V}_o \to \mathcal{D}_o$ and $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ are total functions.
Note further that $\upsilon(\mathcal{E}_\varphi(\mathbf{F}_o)) \equiv \mathtt{F}$ evaluates to $\upsilon(\mathcal{E}_\varphi(\mathbf{T}_o)) \equiv \mathtt{T}$ which we already know. □

Remark 3.26. Note that we only constrain the functional behavior of the values of the logical constants. In particular this does not fully specify these values, since

- $\mathcal{M}$ need not be functional
- there can be more than two truth values.

Definition 3.27 (Properties $\mathfrak{f}$, $\mathfrak{q}$ and $\mathfrak{b}$). Given a Σ-model $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$, we say that $\mathcal{M}$ has **property**

- $\mathfrak{f}$ iff $\mathcal{M}$ is functional.
- $\mathfrak{q}$ iff for all $\alpha \in \mathcal{T}$ there is a function $\mathrm{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$, such that for all $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ holds $\upsilon(\mathrm{q}^\alpha@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$ iff $\mathsf{a} \equiv \mathsf{b}$.
- $\mathfrak{b}$ iff $\mathcal{D}_o$ has at most two elements. Note that $\mathcal{D}_o$ must always have at least the two elements $\mathcal{E}_\varphi(\mathbf{T}_o)$ and $\mathcal{E}_\varphi(\mathbf{F}_o)$ by Lemma 3.25, so we can assume without loss of generality that $\mathcal{D}_o \equiv \{\mathcal{E}_\varphi(\mathbf{F}_o) \equiv \mathtt{F}, \mathcal{E}_\varphi(\mathbf{T}_o) \equiv \mathtt{T}\}$ and that υ is the identity function

Definition 3.28 (Specialized Σ-model Classes). We define special classes of Σ-models depending on the validity of the properties $\mathfrak{f}$, $\mathfrak{q}$ and $\mathfrak{b}$. Thus we obtain $\mathfrak{M}_\mathfrak{q}, \mathfrak{M}_\mathfrak{b}, \mathfrak{M}_\mathfrak{f}, \mathfrak{M}_{\mathfrak{fb}}$ by requiring that the properties specified in the index are valid.

Remark 3.29. We do not introduce $\mathfrak{M}_{\mathfrak{fq}}$, as we will see later (Lemma 3.37) that $\mathfrak{q}$ implies $\mathfrak{f}$ and hence that $\mathfrak{M}_\mathfrak{q} = \mathfrak{M}_{\mathfrak{fq}}$.

As Peter Andrews has noted in [And72a], Leon Henkin unintendedly introduced $\mathfrak{M}_{\mathfrak{fb}}$ in [Hen50] instead of Henkin models in the sense below. A $\mathfrak{M}_{\mathfrak{fb}}$ does not necessarily have property $\mathfrak{q}$ and as Andrews has shown in [And72a], a consequence is, that a $\mathfrak{M}_{\mathfrak{fb}}$ may lack the principle of functional extensionality $\mathrm{EXT}_L^{\alpha\to\beta}$, which he corrected by introducing property $\mathfrak{q}$.

Definition 3.30 (Σ-Henkin models). A functional Σ-model is called a **Σ-Henkin model** ($\mathfrak{H} := \mathfrak{M}_{\mathfrak{qb}}$), iff it has properties $\mathfrak{q}$ and $\mathfrak{b}$. If furthermore, all domains $D_{\alpha\to\beta}$ are full then we call $\mathcal{H}$ a **Σ-standard model ($\mathfrak{ST}$).**

Now let us extend the notion of a quotient structure to Σ-models.

Definition 3.31 (Quotient Σ-model). Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model, $\sim$ a congruence on the corresponding Σ-structure $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$, and $\mathcal{A}/_\sim$ be the quotient Σ-structure of $\mathcal{A} := (\mathcal{D}, @, \mathcal{E})$ modulo $\sim$ as defined in 3.12.

If $\upsilon(\mathbf{A}) \equiv \upsilon(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in wff_o(\Sigma)$ with $\mathbf{A} \sim \mathbf{B}$, then $\sim$ is called a **congruence for** $\mathcal{M}$. Then $\mathcal{M}/_\sim := (\mathcal{D}^\sim, @^\sim, \mathcal{E}^\sim, \upsilon^\sim)$ is called the **quotient Σ-model of $\mathcal{M}$ modulo $\sim$**, if $\upsilon^\sim([\![a]\!]) \equiv \upsilon(a)$ for all $a \in \mathcal{D}_o$.

Remark 3.32. Note the importance of the additional requirement for functional congruence relations stated in 3.31. Without this requirement the quotient Σ-models are not well-defined.

Lemma 3.33. *Let $\mathcal{M}$ be a Σ-model and $\sim$ be a congruence for $\mathcal{M}$, then $\mathcal{M}/_{\sim} \models_\varphi H \subseteq wff_o(\Sigma)$, iff $\mathcal{M} \models_\varphi H$.*

Proof: Let $\mathbf{A}_o \in H$. We have $\upsilon^{\sim}(\mathcal{E}^{\sim}_\varphi(\mathbf{A}_o)) \equiv \upsilon^{\sim}([\![\mathcal{E}_\varphi(\mathbf{A}_o)]\!]) \equiv \upsilon(\mathcal{E}_\varphi(\mathbf{A}_o)) \equiv \mathtt{T}$ since $\mathcal{M} \models H$. □

3.5 Leibniz Equality

Definition 3.34 (Full Extensionality). We call the following formula schemata

$$\begin{array}{lll} \mathrm{EXT}_L^{\alpha\to\beta} & := & \forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}(\forall X_\beta.FX \doteq GX) \Rightarrow F \doteq^\beta G \\ \mathrm{EXT}_L^o & := & \forall A_o.\forall B_o.(A \Leftrightarrow B) \Leftrightarrow A \doteq^o B \end{array}$$

the **axioms of full extensionality for Leibniz equality**; we refer to the first as **axiom of functional extensionality** and to the latter formula as the **extensionality axiom for truth values**. Note that $\mathrm{EXT}_L^{\alpha\to\beta}$ specifies functionality of the relation denoted by the Leibniz formula $\doteq$. We will use the terms functionality and extensionality interchangeably.

Lemma 3.35 (Leibniz Equality in Σ-models). *Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model and φ be an assignment.*

1. *If $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{B})$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B}) \equiv \mathtt{T}$.*
2. *If $\mathcal{M}$ is a $\mathfrak{M}_\mathfrak{b}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^o \mathbf{B})) \equiv \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{B})$.*
3. *If $\mathcal{M}$ is a $\mathfrak{M}_\mathsf{q}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) \equiv \mathtt{T}$, then $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{B})$.*

Proof: Let $\mathsf{a}, \mathsf{b} \in \mathcal{D}_\alpha$ and $\psi := \varphi, [\mathsf{a}/X], [\mathsf{b}/Y]$.

1. We show that $\upsilon(\mathcal{E}_\varphi(\mathbf{Q}^\alpha)@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$ if $\mathsf{a} \equiv \mathsf{b}$, which entails the assertion. By definition $\mathcal{E}_\varphi(\mathbf{Q}^\alpha) \equiv \mathcal{E}_\varphi(\lambda X.\lambda Y.\forall P.PX \Rightarrow PY)$ and thus $\mathcal{E}_\varphi(\mathbf{Q}^\alpha)@\mathsf{a}@\mathsf{b} \equiv \mathcal{E}_\psi(\forall P.PX \Rightarrow PY)$. Now let $\mathsf{r} \in \mathcal{D}_{\alpha\to o}$, then $\upsilon(\mathcal{E}_{\psi,[\mathsf{r}/P]}(PX)) \equiv r@a \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\psi,[\mathsf{r}/P]}(PY)) \equiv \mathsf{r}@\mathsf{b} \equiv \mathsf{r}@\mathsf{a} \equiv \mathtt{T}$, since υ is total and $\mathsf{a} \equiv \mathsf{b}$. So we see that $\upsilon(\mathcal{E}_\varphi(\mathbf{Q})@\mathsf{a}@\mathsf{b}) \equiv \upsilon(\mathcal{E}_{\psi,[\mathsf{r}/P]}(PX \Rightarrow PY)) \equiv \mathtt{T}$ for all $\mathsf{r} \in \mathcal{D}_{\alpha\to o}$, which yields the assertion.
2. First note that by property $\mathfrak{b}$ we have $\mathcal{D}_o \equiv \{\mathtt{T}, \mathtt{F}\}$ and υ is the identity function on $\mathcal{D}_o$. Let us assume that $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^o \mathbf{B})) \equiv \mathcal{E}_\psi(\forall P.P\mathbf{A} \Rightarrow P\mathbf{B}) \equiv \mathtt{T}$ but $\mathcal{E}_\varphi(\mathbf{A}) \not\equiv \mathcal{E}_\varphi(\mathbf{B})$, which means that either $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathtt{T}$ and $\mathcal{E}_\varphi(\mathbf{B}) \equiv \mathtt{F}$ or vice a versa. In the first case we choose a predicate $r := \mathcal{E}_\varphi(\lambda X_o.X_o)$ and get from the first assumption that $\mathcal{E}_{\varphi,[\mathsf{r}/P]}(P\mathbf{A}) \equiv \mathcal{E}_{\varphi,[\mathsf{r}/P]}(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{A}) \equiv \mathtt{F}$ or $\mathcal{E}_{\varphi,[\mathsf{r}/P]}(P\mathbf{B}) \equiv \mathcal{E}_{\varphi,[\mathsf{r}/P]}(\mathbf{B}) \equiv \mathcal{E}_\varphi(\mathbf{B}) \equiv \mathtt{T}$, which gives us the contradiction. Note that P does not occur free in $\mathbf{A}$ or $\mathbf{B}$ by definition of $\doteq$.
The second case is analogous with $r := \mathcal{E}_\varphi(\lambda X_o.\neg X_o)$.
3. We show that if $\upsilon(\mathcal{E}_\varphi(\mathbf{Q}^\alpha)@\mathsf{a}@\mathsf{b}) \equiv \mathtt{T}$ then $\mathsf{a} \equiv \mathsf{b}$, which entails the assertion. Suppose $\mathsf{a} \not\equiv \mathsf{b} \in \mathcal{D}_\alpha$ and $\mathsf{r} \equiv \mathsf{q}^\alpha@\mathsf{a}$ where $\mathsf{q}^\alpha \in \mathcal{D}_{\alpha\to\alpha\to o}$ is the function guaranteed by property q. We know that $\mathsf{q}^\alpha@\mathsf{a}@\mathsf{a} \equiv \mathtt{T}$ and $\mathsf{q}^\alpha@\mathsf{a}@\mathsf{b} \equiv \mathtt{F}$, since $\mathsf{a} \not\equiv \mathsf{b}$ by assumption. Hence $\upsilon(\mathcal{E}_\varphi(\mathbf{Q}^\alpha)@\mathsf{a}@\mathsf{b}) \equiv \upsilon(\mathcal{E}_\psi(\forall P.PX \Rightarrow PY) \equiv \mathtt{F}$ for $\psi := \varphi, [\mathsf{a}/X], [\mathsf{b}/Y]$, since $\upsilon(\mathcal{E}_{\psi,[\mathsf{r}/P]}(PX \Rightarrow PY)) \equiv \mathtt{F}$, as $\upsilon(\mathcal{E}_{\psi,[\mathsf{r}/P]}(PX)) \equiv \mathsf{q}^\alpha@\mathsf{a}@\mathsf{a} \equiv \mathsf{r}@\mathsf{a} \equiv \mathtt{T}$ and $\upsilon(\mathcal{E}_{\psi,[\mathsf{r}/P]}(PY)) \equiv \mathsf{q}^\alpha@\mathsf{a}@\mathsf{b} \equiv \mathsf{r}@\mathsf{b} \equiv \mathtt{F}$. □

Lemma 3.36 (Extensionality in Σ-models).

1. *There exists a $\mathfrak{M}$ which is not functional.*
2. *There exists a $\mathfrak{M}_{\mathfrak{f}\mathfrak{b}}$ for which $\mathrm{EXT}_L^{\alpha\to\beta}$ is not valid.*
3. *There exists a $\mathfrak{M}_\mathsf{q}$ for which EXT_L^o is not valid.*
4. *$\mathrm{EXT}_L^{\alpha\to\beta}$ is valid in $\mathcal{M}$, if $\mathcal{M}$ is a $\mathfrak{M}_\mathsf{q}$.*
5. *EXT_L^o is valid in $\mathcal{M}$, if $\mathcal{M}$ is a $\mathfrak{M}_\mathfrak{b}$.*

As a consequence the following table characterizes the different properties of the introduced semantical structures. If a formula is valid for a certain semantical structure we use a '+' and a '−' otherwise. Each entry is further marked with a justification referring to one of the above statements.

valid in	$\mathfrak{M}/\mathfrak{M}_{\mathfrak{f}}$	$\mathfrak{M}_{\mathfrak{q}}$	$\mathfrak{M}_{\mathfrak{b}}/\mathfrak{M}_{\mathfrak{fb}}$	$\mathfrak{M}_{\mathfrak{qb}}$
$\mathrm{EXT}_L^{\alpha\to\beta}$	$-(2)$	$+(4)$	$-(2)$	$+(4)$
EXT_L^{o}	$-(3)$	$-(3)$	$+(5)$	$+(5)$

Proof: For the proof of 1. note that Σ-models need not to be functional (see also remark 3.26). In the model existence theorem 4.28($\mathfrak{Acc}_{\mathfrak{M}}$) we will later explicitly constructs a functional Σ-model based on the termstructure $\mathcal{TS}(\Sigma)^{\beta}$. For $\mathcal{TS}(\Sigma)^{\beta}$ we already know by remark 3.16 that it is not functional.

For the proof of 2. we refer to [And72a], where Andrews constructs a functional Σ-Model (actually a $\mathfrak{M}_{\mathfrak{fb}}$) which lacks the principle of functional extensionality of Leibnizequality.

For 3. note that EXT_L^o can only be valid if $\mathcal{D}_o = \{o, \iota\}$, which is not required for $\mathfrak{M}_{\mathfrak{q}}$'s. For a concrete example of a $\mathfrak{M}_{\mathfrak{q}}$ which lacks EXT_L^o see 4.28($\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{q}}}$).

Next we consider 4.: Let $\psi := \varphi, [\mathsf{f}/F], [\mathsf{g}/G]$. From $\mathcal{V}_\psi(\forall A_\alpha . FA \doteq GA) \equiv \mathtt{T}$ we get that for all $\mathsf{a} \in \mathcal{D}_\alpha$ $\mathcal{V}_{\psi,[\mathsf{a}/A]}(FA \doteq GA) \equiv \mathtt{T}$. By lemma 3.35(3) we can conclude that $\mathcal{E}_{\psi,[\mathsf{a}/A]}(FA) \equiv \mathcal{E}_{\psi,[\mathsf{a}/A]}(GA)$ for all $\mathsf{a} \in \mathcal{D}_\alpha$ and hence $\mathcal{E}_{\psi,[\mathsf{a}/A]}(F)@\mathcal{E}_{\psi,[\mathsf{a}/A]}(A) \equiv \mathcal{E}_{\psi,[\mathsf{a}/A]}(G)@\mathcal{E}_{\psi,[\mathsf{a}/A]}(A)$. The application of functionality leads to $\mathcal{E}_\psi(F) \equiv \mathcal{E}_\psi(G)$ which finally gives us that $\mathcal{V}_\psi(F \doteq^{\alpha\to\beta} G) \equiv \mathtt{T}$ with lemma 3.35(1).

And finally in 5. +we have that for all $\mathsf{a}, \mathsf{b} \in \mathcal{D}_o$ and all assignments φ $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/A][b/B]}(A \Leftrightarrow B)) \equiv \mathtt{T}$, iff $\upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/A][\mathsf{b}/B]}(A)) \equiv \upsilon(\mathcal{E}_{\varphi,[\mathsf{a}/A][\mathsf{b}/B]}(B))$. From $\mathfrak{b}$ we further know that υ is the identity function and hence this statement is valid, iff $\mathcal{E}_{\varphi,[\mathsf{a}/A][\mathsf{b}/B]}(A) \equiv \mathcal{E}_{\varphi,[\mathsf{a}/A][\mathsf{b}/B]}(B)$ from which we get the assertion by lemma 3.35(1).

□

We are now in a good position to prove the assertion that property $\mathfrak{q}$ implies property $\mathfrak{f}$ stated in remark 3.29. Thus the next lemma shows that requiring property $\mathfrak{q}$ automatically introduces $\mathfrak{f}$ and hence there cannot be a distinction between $\mathfrak{M}_{\mathfrak{q}}$'s and $\mathfrak{M}_{\mathfrak{fq}}$'s.

Lemma 3.37 ($\mathfrak{q}$ implies $\mathfrak{f}$). *Let $\mathcal{M}$ be a Σ-model with property $\mathfrak{q}$. Then $\mathcal{M}$ has property $\mathfrak{f}$.*

Proof: Let $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a $\mathfrak{M}_{\mathfrak{q}}$ and let $\varphi := \psi, [f/F], [g/G]$ for an arbitrary assignment ψ and arbitrary $f, g \in \mathcal{D}_{\alpha\to\beta}$. We show that if for all $a \in \mathcal{D}_\alpha$ holds that $\mathcal{E}_{\varphi,[a/X]}(FX) \equiv \mathcal{E}_{\varphi,[a/X]}(GX)$ then $\mathcal{E}_\varphi(F) \equiv \mathcal{E}_\varphi(G)$, which entails the assertion. From $\mathcal{E}_{\varphi,[a/X]}(FX) \equiv \mathcal{E}_{\varphi,[a/X]}(GX)$ for all $a \in \mathcal{D}_\alpha$ we get with Lemma 3.35(1) that $\upsilon(\mathcal{E}_{\varphi,[a/X]}(FX \doteq GX)) \equiv \mathtt{T}$ for all $a \in \mathcal{D}_\alpha$ and hence that $\upsilon(\mathcal{E}_\varphi(\forall X_\alpha . FX \doteq GX)) \equiv \mathtt{T}$. We can apply $\mathrm{EXT}_L^{\alpha\to\beta}$, which is valid in $\mathcal{M}$ by 3.36(4), and thus we get that $\upsilon(\mathcal{E}_\varphi(F \doteq G)) \equiv \mathtt{T}$. Now the conclusion follows by Lemma 3.35(3). □

Next we discuss the role of Leibniz equality within the different semantic structures.

Theorem 3.38 (Properties of Leibniz Equality). *Let $\mathcal{M}$ be a Σ-model. For all assignments φ and all terms $\mathbf{A}, \mathbf{B}, \mathbf{C} \in wff_\alpha(\Sigma)$ and $\mathbf{F}, \mathbf{G} \in wff_{\alpha\to\beta}(\Sigma)$ we have:*

$\mathfrak{M}$ *$\mathcal{E}_\varphi(\doteq^\alpha)$ is an equivalence relation on $\mathcal{D}_\alpha$ with respect to υ. In particular:*

- **re** $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{A})) \equiv \mathtt{T}$.
- **sy** *If $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) \equiv \mathtt{T}$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{B} \doteq^\alpha \mathbf{A})) \equiv \mathtt{T}$.*
- **tr** *If $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) \equiv \mathtt{T}$ and $\upsilon(\mathcal{E}_\varphi(\mathbf{B} \doteq^\alpha \mathbf{C})) \equiv \mathtt{T}$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{C})) \equiv \mathtt{T}$.*

$\mathfrak{M}_{\mathfrak{f}}$ *If $\mathcal{M}$ is a $\mathfrak{M}_{\mathfrak{f}}$, then $\mathcal{E}_\varphi(\doteq^\alpha)$ is a congruence relation on $\mathcal{D}_\alpha$ with respect to υ. In particular:*

- **co** *If $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) \equiv \mathtt{T}$ then $\upsilon(\mathcal{E}_\varphi(\mathbf{FA} \doteq^\beta \mathbf{FB})) \equiv \mathtt{T}$.*

$\mathfrak{M}_{\mathfrak{q}}$ *If $\mathcal{M}$ is a $\mathfrak{M}_{\mathfrak{q}}$, then $\mathcal{E}_\varphi(\doteq^\alpha)$ is a functional congruence relation on $\mathcal{D}_\alpha$ with respect to υ. In particular:*

- **fu** $\upsilon(\mathcal{E}_\varphi(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})) \equiv \mathtt{T}$ *if* $\upsilon(\mathcal{E}_\varphi(\mathbf{FA} \doteq^\beta \mathbf{FA})) \equiv \mathtt{T}$ *for all* $\mathbf{A} \in wff_\alpha$.

$\mathfrak{M}_{\mathfrak{qb}}$ *If $\mathcal{M}$ is a $\mathfrak{H}$, then $\mathcal{E}_\varphi(\doteq^\alpha)$ is the equality relation on $\mathcal{D}_\alpha$.*

Proof:

$\mathfrak{M}$ **re:** $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{A})) \equiv \mathtt{T}$, iff for all $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$ we have $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P\mathbf{A})) \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P\mathbf{A})) \equiv \mathtt{T}$ which is obvious since υ is total and surjective.

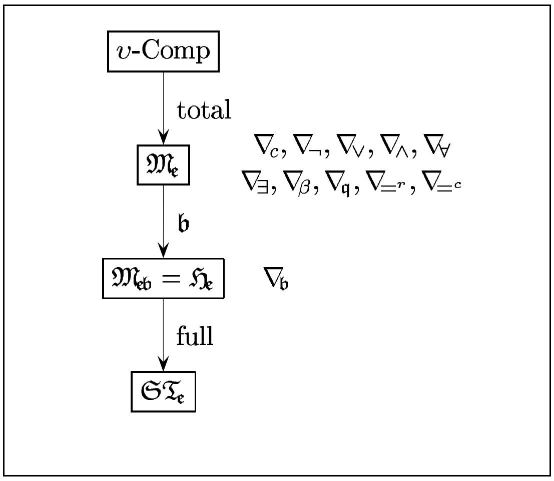

Figure 2: The landscape of Higher-Order Semantics with primitive equality

sy: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq^\alpha \mathbf{B})) \equiv \mathtt{T}$ but $\upsilon(\mathcal{E}_\varphi(\mathbf{B} \doteq^\alpha \mathbf{A})) \equiv \mathtt{F}$. From the latter we get that $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P\mathbf{B})) \equiv \mathtt{T}$ and $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P\mathbf{A})) \equiv \mathtt{F}$ for some $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$. Without loss of generality, let $\mathsf{p} := \mathcal{E}_\varphi(V)$ for a fresh variable $V \in \Sigma_{\alpha\to o}$. From the former assumption we know that for all $\mathsf{q} \in \mathcal{D}_{\alpha\to o}$ holds $\upsilon(\mathcal{E}_{\varphi,[\mathsf{q}/P]}(P\mathbf{A})) \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi,[\mathsf{q}/P]}(P\mathbf{B})) \equiv \mathtt{T}$ and hence $\upsilon(\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\lambda X.VX)/P]}(P\mathbf{A})) \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\lambda X.VX)/P]}(P\mathbf{B})) \equiv \mathtt{T}$ which is equivalent with $\upsilon(\mathcal{E}_{\varphi,[p/P]}(P\mathbf{A})) \equiv \mathtt{T}$ or $\upsilon(\mathcal{E}_{\varphi,[p/P]}(P\mathbf{B})) \equiv \mathtt{F}$ and contradicts the latter assumption.
tr: Similar to **sy**.

$\mathfrak{M}_{\mathfrak{f}}$ **co**: Suppose $\upsilon(\mathcal{E}_\varphi(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})) \equiv \mathtt{T}$ but $\upsilon(\mathcal{E}_\varphi(\mathbf{FA} \doteq^\alpha \mathbf{GA})) \equiv \mathtt{F}$. From the latter we get that $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P(\mathbf{FA}))) \equiv \mathtt{T}$ and $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P(\mathbf{GA}))) \equiv \mathtt{F}$ for some $\mathsf{p} \in \mathcal{D}_{\alpha\to o}$. Without loss of generality let $\mathsf{p} := \mathcal{E}_\varphi(V)$ for a fresh variable $V \in \mathcal{V}_{\alpha\to o}$. From the former assumption we know that for all $\mathsf{q} \in \mathcal{D}_{\alpha\to o}$ holds $\upsilon(\mathcal{E}_{\varphi,[\mathsf{q}/Q]}(Q\mathbf{F})) \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi,[\mathsf{q}/Q]}(Q\mathbf{G})) \equiv \mathtt{T}$ and hence $\upsilon(\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\lambda X.V(XA))/P]}(P\mathbf{F})) \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi,[\mathcal{E}_\varphi(\lambda X.V(XA))/P]}(P\mathbf{F})) \equiv \mathtt{T}$ which is equivalent with $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P(\mathbf{FA}))) \equiv \mathtt{F}$ or $\upsilon(\mathcal{E}_{\varphi,[\mathsf{p}/P]}(P(\mathbf{GA}))) \equiv \mathtt{T}$ and contradicts the former assumption.

$\mathfrak{M}_{\mathfrak{q}}$ **fu**: A direct consequence of lemma 3.36(4).

$\mathfrak{M}_{\mathfrak{qb}}$ By property $\mathfrak{b}$ we know that υ is the identity relation on $\mathcal{D}_o$ and thus we have that $\doteq$ denotes a relation for which the principles reflexivity, symmetry, transitivity, congruence and functionality hold. Hence $\doteq$ denotes the equality relation. □

3.6 Primitive Equality

The situation of higher-order semantics becomes much simpler if we introduce equality as a primitive logical constant $=$ in Σ, which we will assume for the rest of this section. Since $=$ is logical, we have to specialize the notion of Σ-valuation (cf. 3.24) by requiring that $\upsilon(\mathcal{E}(=^\alpha)@a@b) \equiv \mathtt{T}$, iff $a \equiv b$. In this case, we call υ a **Σ-valuation with equality**.

Furthermore, we say that a Σ-model $\mathcal{M}$ has **property $\mathfrak{e}$**, iff for all $a, b \in \mathcal{D}_\alpha$ we have $\upsilon(\mathcal{E}(=^\alpha)@a@b) \equiv \mathtt{T}$, iff $a \equiv b$.

A (functional) Σ-model, which has property $\mathfrak{e}$ is called a (functional) **Σ-model with full equality ($\mathfrak{M}_{\mathfrak{e}}$)** and a functional one with additional property $\mathfrak{b}$ is called a **Σ-Henkin model with full equality ($\mathfrak{M}_{\mathfrak{eb}}$)**.

Clearly, property $\mathfrak{e}$ entails property $\mathfrak{q}$, since $\mathcal{E}(=^\alpha)$ is the function required in property $\mathfrak{q}$. And we already know that property $\mathfrak{q}$ implies property $\mathfrak{f}$, it is easy to see that the landscape of higher-order semantics from figure 1 collapses to one in figure 2.

The connection between property $\mathfrak{q}$ and $\mathfrak{e}$ is already discussed in [And72a]. Andrews concludes that it seems natural to require the existence of logical connectives $=^\alpha$ in the signature, if one is interested in extensionality. In this paper we are especially interested to shed some light on both: in extensionality of Leibniz equality in case $=^\alpha \notin \Sigma$ and in extensionality of Leibniz equality and/or primitive equality in case $=^\alpha \in \Sigma$.

Definition 3.39 (Extensionality). Analogous to the extensionality Axioms for Leibniz equality, we can define such for primitive equality.

$$\begin{array}{lll} \mathrm{EXT}^{\alpha\to\beta} & := & \forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}.(\forall X_\beta.FX = GX) \Rightarrow F =^\beta G \\ \mathrm{EXT}^{o} & := & \forall A_o.\forall B_o.(A \Leftrightarrow B) \Leftrightarrow A =^o B \end{array}$$

the **axioms of full extensionality for primitive equality.**

The following lemma shows that in a Σ-model with full equality the denotations of primitive equations and corresponding Leibniz equations are identical modulo υ.

Lemma 3.40 (Primitive and Leibniz equality). *If $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon) \in \mathfrak{M}_\mathfrak{e}$, then $\upsilon(\mathcal{E}_\varphi(\mathbf{A} = \mathbf{B})) \equiv \upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq \mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in wff(\Sigma)$.*

Proof: By lemma 3.35(3) we have $\upsilon(\mathcal{E}_\varphi(\mathbf{A} \doteq \mathbf{B})) \equiv \mathtt{T}$, iff $\mathcal{E}_\varphi(\mathbf{A}) \equiv \mathcal{E}_\varphi(\mathbf{B})$, since $\mathfrak{M}_\mathfrak{e} \subseteq \mathfrak{M}_\mathfrak{q}$. By property $\mathfrak{e}$ this is equivalent with $\upsilon(\mathcal{E}_\varphi(\mathbf{A} = \mathbf{B})) \equiv \mathtt{T}$. □

Lemma 3.41 (Extensionality in Σ-models with full equality).

1. *There exists a $\mathfrak{M}_\mathfrak{e}$ which is not functional.*
2. *There exists a $\mathfrak{M}_\mathfrak{e}$ for which EXT^o and EXT^o_L are not valid.*
3. *$\mathrm{EXT}^{\alpha\to\beta}$ and $\mathrm{EXT}^{\alpha\to\beta}_L$ are valid in $\mathcal{M}$, if $\mathcal{M}$ is a $\mathfrak{M}_\mathfrak{e}$*
4. *EXT^o and EXT^o is valid in $\mathcal{M}$, if $\mathcal{M}$ is a $\mathfrak{M}_{\mathfrak{eb}}$.*

Thus we can extend the table in Lemma 3.36 to the following one:

valid in	$\mathfrak{M}_\mathfrak{e}$	$\mathfrak{M}_{\mathfrak{eb}}$
$\mathrm{EXT}^{\alpha\to\beta}_L, \mathrm{EXT}^{\alpha\to\beta}$	+	+
$\mathrm{EXT}^o_L, \mathrm{EXT}^o$	−	+

Proof:

1. The argumentation is analogous to 3.36(1) and a concrete example of a non-functional $\mathfrak{M}_\mathfrak{e}$ is given in 4.28($\mathfrak{M}_\mathfrak{e}$).
2. The argumentation is analogous to 3.36(2) and 3.36(3). A concrete example of a $\mathfrak{M}_\mathfrak{e}$ which lacks EXT^o_L and EXT^o is provided by 4.28($\mathfrak{M}_\mathfrak{e}$).
3. Note that the only crucial points in the proof of 3.36(4) are functionality, which is given here as well, and the application of lemmata 3.35(1) and 3.35(3). Since a $\mathfrak{M}_\mathfrak{e}$ is also a $\mathfrak{M}_\mathfrak{q}$ both lemmata are applicable here as well and thus for $\doteq$ we get the statement immediately. For $=$ the statement can be proven analogously to 3.36(4) using property $\mathfrak{e}$ instead of the lemmata 3.35(1) and 3.35(3).
4. In the proof of 3.36(5) the only crucial parts are the usage of property $\mathfrak{b}$ and lemma 3.35(1). Again for $\doteq$ there is nothing to show, since a $\mathfrak{M}_{\mathfrak{eb}}$ is also a $\mathfrak{M}_{\mathfrak{qb}}$. The statement for $=$ can be proven analogously with property $\mathfrak{e}$ instead of lemma 3.35(1). □

Theorem 3.42. *Let $\mathcal{M} \in \mathfrak{M}_\mathfrak{e}$, then $\mathcal{E}_\varphi(\doteq^\alpha)$ and $\mathcal{E}_\varphi(=^\alpha)$ are equivalence relations on $\mathcal{D}_\alpha$ with respect to υ for all assignments φ. If $\mathcal{M}$ is a $\mathfrak{M}_{\mathfrak{eb}}$, then $\mathcal{E}_\varphi(\doteq^\alpha) = \mathcal{E}_\varphi(=^\alpha)$ is the equality relation on $\mathcal{D}_\alpha$.*

Proof: Note that for $\doteq$ the proofs are provided by lemma 3.38, since $\mathfrak{M}_\mathfrak{e} \subseteq \mathfrak{M}_\mathfrak{q}$ and $\mathfrak{M}_{\mathfrak{eb}} \subseteq \mathfrak{M}_{\mathfrak{qb}}$. Thus it remains to verify the statements for $=$. Let $\mathcal{M} \in \mathfrak{M}_\mathfrak{e}$, then reflexivity, symmetry and transitivity follow from their $\doteq$-counterparts by lemma 3.40. Functionality is a direct consequence of lemma 3.41(3) and **co** follows from the functionality of a $\mathfrak{M}_\mathfrak{e}$ together with property $\mathfrak{e}$.

If $\mathcal{M} \in \mathfrak{M}_{\mathfrak{eb}}$, then the argumentation for both, $\doteq$ and $=$, is analogous to 3.41($\mathfrak{M}_{\mathfrak{qb}}$): By property $\mathfrak{b}$ we know that υ is the identity relation on $\mathcal{D}_o$ and thus we have that $\doteq$ and $=$ denote relations for which the principles reflexivity, symmetry, transitivity, congruence and functionality hold. Hence both, $\doteq$ and $=$, denote the equality relation, since the fact that therer are only two truth values does not leave any room for other relations with these properties. $\square$

4 Model Existence Theorems

In this section we introduce the model existence theorems for the different semantical notions discussed in section 3. These theorems have the following form, where $* \in \{\mathfrak{M}, \mathfrak{M}_{\mathfrak{b}}, \mathfrak{M}_{\mathfrak{f}}, \mathfrak{M}_{\mathfrak{q}}, \mathfrak{M}_{\mathfrak{fb}}, \mathfrak{M}_{\mathfrak{qb}}\ \mathfrak{M}_{\mathfrak{e}}, \mathfrak{M}_{\mathfrak{eb}}\}$:

> **Theorem (Model Existence):** *For a given abstract consistency class $\mathfrak{Acc}_*$ and a set $H \in \mathfrak{Acc}_*$ there is a $*$-model of H.*

The most important tools used in the proofs of the model existence theorems are the so-called Σ-Hintikka sets. These sets are maximal elements in abstract consistency classes, and allow computations that resemble those in the considered semantical structures (e.g. Σ-Henkin models). These allow to construct $*$-valuations for the term structures that turn those into $*$-models.

The key step in the proof of the model existence theorems is an extension lemma, which guarantees a Σ-Hintikka set $\mathcal{H}$ for any set H of sentences in Γ_Σ. With this, the proofs for the model existence theorems are uniform.

4.1 Abstract Consistency

Let us now review a few technicalities that we will need for the proofs of the model existence theorems.

Definition 4.1 (Compactness). Let $\mathcal{C}$ be a class of sets.

1. $\mathcal{C}$ is called **closed under subsets**, iff for all sets S and T the following condition holds: if $S \subseteq T$ and $T \in \mathcal{C}$, then $S \in \mathcal{C}$.
2. $\mathcal{C}$ is called **compact** iff for every set S the following condition holds: $S \in \mathcal{C}$, iff every finite subset of S is a member of $\mathcal{C}$.

Lemma 4.2. *If $\mathcal{C}$ is compact, then $\mathcal{C}$ is closed under subsets.*

Proof: Suppose $S \subseteq T$ and $T \in \mathcal{C}$. Every finite subset A of S is a finite subset of T, and since $\mathcal{C}$ is compact, we know that $A \in \mathcal{C}$. Thus $S \in \mathcal{C}$. $\square$

Definition 4.3 (Sufficiently Pure). Let Σ be a signature and $\mathcal{T}$ be a set of Σ-sentences. $\mathcal{T}$ is called **sufficiently Σ-pure**, iff for each type α there is a set of constants $\mathcal{P}_\alpha \subseteq \Sigma_\alpha$ with equal cardinality to $\mathit{wff}_\alpha(\Sigma)$, such that the elements of $\mathcal{P}$ do not occur in $\mathcal{T}$.

We will always presuppose that sets of sets of sentences are sufficiently Σ-pure in order to have enough witness constants. This can be obtained in practice by enriching the signature with spurious constants. Another way would be to use specially marked variables (which may never be instantiated) as in [Koh94b].

Definition 4.4 (Properties for Abstract Consistency Classes). Let Γ_Σ be a class of sets of Σ-sentences. We need the following conditions, where $\mathbf{A}, \mathbf{B} \in \mathit{cwff}_o(\Sigma)$ and $\mathbf{F}, \mathbf{G} \in \mathit{cwff}_{\alpha\to\beta}(\Sigma)$:[10]

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

[10]In the following we will use $\varphi * \mathbf{A}$ as an abbreviation for $\varphi \cup \{A\}$.

∇_β If $\mathbf{A} \in \Phi$ and $\mathbf{B}$ is the β-normal form of $\mathbf{A}$, then $\mathbf{B} * \Phi \in \Gamma_\Sigma$.
$\nabla_\mathfrak{f}$ If $\mathbf{A} \in \Phi$ and $\mathbf{B}$ is the $\beta\eta$-normal form of $\mathbf{A}$, then $\mathbf{B} * \Phi \in \Gamma_\Sigma$.
$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.
$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi \cup \{\neg\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$.
$\nabla_\forall$ If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{FW} \in \Gamma_\Sigma$ for each $\mathbf{W} \in cwff_\alpha(\Sigma)$.
$\nabla_\exists$ If $\neg\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any constant $w \in \Sigma_\alpha$, which does not occur in Φ.
$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi \cup \{\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$ or $\Phi \cup \{\neg\mathbf{A}, \mathbf{B}\} \in \Gamma_\Sigma$.
$\nabla_\mathfrak{q}$ If $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \Phi$, then $\Phi * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w) \in \Gamma_\Sigma$ for any constant $w \in \Sigma_\alpha$, which does not occur in Φ.

(Additional abstract consistency conditions for primitive equality will be introduced later in section 4.3.)

Remark 4.5. Note that for the connectives $\vee, \Pi^\alpha$ there are two conditions – a positive and a negative one – given in the definition above, namely $\nabla_\vee/\nabla_\wedge$ for $\vee$ and $\nabla_\forall/\nabla_\exists$ for Π^α. For $\doteq^o$ and $\doteq^{\alpha\to\beta}$ the situation is different, as we need only conditions for the negative cases. The positive cases can be inferred at level of Hintikka sets by expanding the Leibniz definition of equality (see the proofs of $\overline{\nabla}_{\mathfrak{q}'}$ in lemma 4.15 and $\overline{\nabla}_{\mathfrak{b}'}$ in lemma 4.17).

Definition 4.6 (Abstract Consistency Classes). Let Σ be a signature Γ_Σ be a class of sets of Σ-propositions. Using the properties from the previous definition we introduce the following abstract consistency classes:

$\mathfrak{Acc}_\mathfrak{M}$ If $\nabla_c, \nabla_\neg, \nabla_\beta, \nabla_\vee, \nabla_\wedge, \nabla_\forall$ and $\nabla_\exists$ are valid for Γ_Σ, then Γ_Σ is called an **abstract consistency class for Σ-models ($\mathfrak{Acc}_\mathfrak{M}$).**

Based upon this definition we introduce the following specialized abstract consistency classes: $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{b}}, \mathfrak{Acc}_{\mathfrak{M}_\mathfrak{f}}, \mathfrak{Acc}_{\mathfrak{M}_\mathfrak{q}}, \mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{fb}}}, \mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}} (= \mathfrak{Acc}_\mathfrak{H})$, where we indicate by indices which additional properties from $\{\nabla_\mathfrak{f}, \nabla_\mathfrak{q}, \nabla_\mathfrak{b}\}$ are required.

Sometimes we do not want to differentiate between the particular notions above. In this cases we simply speak of an **abstract consistency class**, with which we refer to an arbitrary but one in $\{\mathfrak{Acc}_\mathfrak{M}, \mathfrak{Acc}_{\mathfrak{M}_\mathfrak{b}}, \mathfrak{Acc}_{\mathfrak{M}_\mathfrak{f}}, \mathfrak{Acc}_{\mathfrak{M}_\mathfrak{q}}, \mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{fb}}}, \mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}\}$.

Remark 4.7. Note that $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{f}}$ corresponds to the abstract consistency property discussed by Andrews in [And71]. The only (technical) difference is that Andrews does not consider α-conversion as built-into the logic but needs a condition similar to ∇_β that requires α-standardized forms to be abstract consistent.

Lemma 4.8 (Non-atomic consistency). *Let Γ_Σ be an abstract consistency class and $\mathbf{A} \in cwff_o(\Sigma)$, then for all $\Phi \in \Gamma_\Sigma$ we have $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.*

Proof: Let $\mathbf{A} \in wff_o(\Sigma)$ and $\Phi \in \Gamma_\Sigma$, such that $\mathbf{A} \in \Phi$. By ∇_β, we can assume that $\mathbf{A}$ is a β-normal form. So we prove the assertion by an induction over the structure of $\mathbf{A}$.

If $\mathbf{A}$ is atomic, then we get the assertion immediately by ∇_c. If $\mathbf{A}$ is not atomic, then its head must be a logical constant, therefore we can proceed by a case-analysis over the connectives and quantifiers.

Suppose $\mathbf{A}$ has the form $\neg\mathbf{B}$ and $\{\neg\mathbf{B}, \neg\neg\mathbf{B}\} \subseteq \Phi$. By $\nabla_\neg$ we know that $\{\neg\mathbf{B}, \mathbf{B}\} \cup \Phi \in \Gamma_\Sigma$ which contradicts the induction hypotheses. Now suppose $\mathbf{A}$ has the form $\mathbf{B} \vee \mathbf{C}$ and $\{\mathbf{B} \vee \mathbf{C}, \neg(\mathbf{B} \vee \mathbf{C})\} \subseteq \Phi$. By $\nabla_\vee$, $\nabla_\wedge$ we know that $\{\mathbf{B} \vee \mathbf{C}, \neg(\mathbf{B} \vee \mathbf{C}), \mathbf{B}, \neg\mathbf{B}, \neg\mathbf{C}\} \cup \Phi \in \Gamma_\Sigma$ or $\{\mathbf{B} \vee \mathbf{C}, \neg(\mathbf{B} \vee \mathbf{C}), \mathbf{C}, \neg\mathbf{B}, \neg\mathbf{C}\} \cup \Phi \in \Gamma_\Sigma$. In both cases the contradiction is given by the induction hypotheses. Suppose $\mathbf{A}$ has the form $\Pi(\lambda X.\mathbf{B})$ and $\{\Pi(\lambda X.\mathbf{B}), \neg\Pi(\lambda X.\mathbf{B})\} \subseteq \Phi$. By $\nabla_\exists$, $\nabla_\forall$ and ∇_β we know that $\{\Pi(\lambda Y.\mathbf{B}), \neg\Pi(\lambda Y.\mathbf{B}), [\mathbf{W}/Y]\mathbf{B}, \neg[\mathbf{W}/Y]\mathbf{B}\} \cup \Phi \in \Gamma_\Sigma$ which contradicts again the induction hypotheses. □

In contrast to [And71], we work with saturated abstract consistency classes in order to obtain total Σ-valuations, which makes the proofs of the model existence theorem much simpler and e.g. yield much more natural models.

Definition 4.9 (Saturated). We call an abstract consistency class Γ_Σ **atomically saturated**, iff for all $\Phi \in \Gamma_\Sigma$ and for all atomic sentences $\mathbf{A} \in cwff_o(\Sigma)$, we have $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \neg\mathbf{A} \in \Gamma_\Sigma$. If this property holds for all sentences $\mathbf{A} \in cwff_o(\Sigma)$, then we call Γ_Σ **saturated.**

Remark 4.10. Clearly, not all abstract consistency classes are saturated, since the empty set is one that is not, even if Σ is empty.

In the definition of abstract consistency class, we only had to require atomic consistency, i.e. that there are no atomic propositions that contradict each other in one abstract consistent set, to ensure consistency (see 4.8). The authors conjecture that a similar theorem can be proven for saturatedness:

> **Conjecture:** *Let Γ_Σ be an atomic saturated abstract consistency class. Then there exists an saturated abstract consistency class Γ'_Σ, with Γ_Σ is a subclass of Γ'_Σ.*

Such a result would be of practical importance, as it allows to reduce the problem of proving saturatedness of a given calculus to proving atomic saturatedness.

Lemma 4.11. *Let Γ_Σ be a saturated abstract consistency class, $\Phi \in \Gamma_\Sigma$ and $\mathbf{A}$ an atomic sentence. Then $\Phi * (\mathbf{A} \lor \neg\mathbf{A}) \in \Gamma_\Sigma$.*

Proof: Since Γ_Σ is saturated and $\Phi \in \Gamma_\Sigma$, we must have $\Phi * (\mathbf{A} \lor \neg\mathbf{A}) \in \Gamma_\Sigma$ or $\Phi * \neg(\mathbf{A} \lor \neg\mathbf{A}) \in \Gamma_\Sigma$. We prove the assertion by refuting the second alternative. If $\Phi * \neg(\mathbf{A} \lor \neg\mathbf{A}) \in \Gamma_\Sigma$, then $\Phi \cup \{\neg(\mathbf{A} \lor \neg\mathbf{A}), \neg\mathbf{A}, \neg\neg\mathbf{A}, \mathbf{A}\} \in \Gamma_\Sigma$ by $\nabla_\land$ and $\nabla_\neg$. Since $\mathbf{A}$ is an atomic sentence we get a contradiction with lemma 4.8. □

Lemma 4.12 (Compactness of abstract consistency classes). *For each abstract consistency class Γ_Σ exists an abstract consistency class Γ'_Σ of the same type, such that $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$, and Γ'_Σ is compact. Furthermore Γ_Σ is saturated, iff Γ'_Σ is.*

Proof: (following and extending [And86], proposition no. 2506)
We choose $\Gamma'_\Sigma := \{\Phi \subseteq cwff_o(\Sigma) \mid \text{every finite subset of } \Phi \text{ is in } \Gamma_\Sigma\}$. Now suppose that $\Phi \in \Gamma_\Sigma$. Γ_Σ is closed under subsets, so every finite subset of Φ is in Γ_Σ and thus $\Phi \in \Gamma'_\Sigma$. Hence $\Gamma_\Sigma \subseteq \Gamma'_\Sigma$.

Next let us show that each Γ'_Σ is compact. Suppose $\Phi \in \Gamma'_\Sigma$ and Ψ is an arbitrary finite subset of Φ. By definition of Γ'_Σ all finite subsets of Φ are in Γ_Σ and therefore $\Psi \in \Gamma'_\Sigma$. Thus all finite subsets of Φ are in Γ'_Σ whenever Φ is in Γ'_Σ. On the other hand, suppose all finite subsets of Φ are in Γ'_Σ. Then by the definition of Γ'_Σ the finite subsets of Φ are also in Γ_Σ, so $\Phi \in \Gamma'_\Sigma$. Thus Γ'_Σ is compact.

Next we show that if Γ_Σ satisfies ∇_*, then Γ'_Σ satisfies ∇_*, by considering the cases of definition 4.6. First note that by lemma 4.2 we have that Γ'_Σ is closed under subsets.

- ∇_c Let $\Phi \in \Gamma'_\Sigma$ and suppose there is an atom $\mathbf{A}$ such that $\{\mathbf{A}, \neg\mathbf{A}\} \subseteq \Phi$. Then $\{\mathbf{A}, \neg\mathbf{A}\} \in \Gamma_\Sigma$ contradicting ∇_c.
- $\nabla_\neg$ Let $\Phi \in \Gamma'_\Sigma$, $\neg\neg\mathbf{A} \in \Phi$, Ψ be any finite subset of $\Phi * \mathbf{A}$ and $\Theta := (\Psi \setminus \{\mathbf{A}\}) * \neg\neg\mathbf{A}$. Θ is a finite subset of Φ, so $\Theta \in \Gamma_\Sigma$. Since Γ_Σ is an abstract consistency class and $\neg\neg\mathbf{A} \in \Theta$, we get $\Theta * \mathbf{A} \in \Gamma_\Sigma$ by $\nabla_\neg$. We know that $\Psi \subseteq \Theta * \mathbf{A}$ and Γ_Σ is closed under subsets, so $\Psi \in \Gamma_\Sigma$. Thus every finite subset Ψ of $\Phi * \mathbf{A}$ is in Γ_Σ and therefore by definition $\Phi * \mathbf{A} \in \Gamma'_\Sigma$.
- $\nabla_\beta, \nabla_\mathfrak{f}, \nabla_\lor, \nabla_\land, \nabla_\forall, \nabla_\exists$ Analogous to $\nabla_\neg$.
- $\nabla_\mathfrak{q}$ Let $\Phi \in \Gamma'_\Sigma$, $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \Phi$ and Ψ be any finite subset of $\Phi * \neg(\mathbf{FW} \doteq \mathbf{GW})$. We show that $\Psi \in \Gamma_\Sigma$. Clearly $\Theta := (\Psi \setminus \{\neg(\mathbf{FW} \doteq \mathbf{GW})\}) * \neg(\mathbf{F} \doteq \mathbf{G})$ is a finite subset of Φ and therefore $\Theta \in \Gamma_\Sigma$. Since Γ_Σ satisfies $\nabla_\mathfrak{q}$ and $\neg(\mathbf{F} \doteq \mathbf{G}) \in \Theta$, we have $\Theta * \neg(\mathbf{FW} \doteq \mathbf{GW}) \in \Gamma_\Sigma$ by $\nabla_\mathfrak{q}$. Furthermore, $\Psi \subseteq \Theta * \neg(\mathbf{FW} \doteq \mathbf{GW})$ and Γ_Σ is closed under subsets, so $\Psi \in \Gamma_\Sigma$. Thus every finite subset Ψ of $\Phi * \neg(\mathbf{FW} \doteq \mathbf{GW})$ is in Γ_Σ, therefore by definition we have $\Phi * \neg(\mathbf{FW} \doteq \mathbf{GW}) \in \Gamma'_\Sigma$.
- $\nabla_\mathfrak{b}$ Let $\Phi \in \Gamma'_\Sigma$ with $\neg(\mathbf{A} \doteq \mathbf{B}) \in \Phi$ but $\Phi \cup \{\mathbf{A}, \neg\mathbf{B}\} \notin \Phi$ and $\Phi \cup \{\neg\mathbf{A}, \mathbf{B}\} \notin \Phi$. Then there exists finite subsets Φ_1 and Φ_2 of Φ such that $\Phi_1 * \{\mathbf{A}, \neg\mathbf{B}\} \notin \Gamma_\Sigma$ and $\Phi_2 * \{\neg\mathbf{A}, \mathbf{B}\} \notin$

Γ_Σ. Now we choose $\Phi_3 := \Phi_1 \cup \Phi_2 * \neg(\mathbf{A} \doteq \mathbf{B})$. Obviously Φ_3 is a finite subset of Φ and therefore $\Phi_3 \in \Gamma_\Sigma$. Since Γ_Σ satisfies $\nabla_{\mathfrak{b}}$, we have that $\Phi_3 \cup \{\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$ or $\Phi_3 \cup \{\neg\mathbf{A}, \mathbf{B}\} \in \Gamma_\Sigma$. From this and the fact that extensional abstract consistency classes are closed under subsets we get that $\Phi_1 \cup \{\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$ or $\Phi_2 \cup \{\neg\mathbf{A}, \mathbf{B}\} \in \Gamma_\Sigma$, which contradicts our assumption.

For the proof that Γ'_Σ is saturated, let $\Phi \in \Gamma'_\Sigma$, but neither $\Phi * \mathbf{A}$ nor $\Phi * \neg\mathbf{A}$ be in Γ'_Σ. Then there are finite subsets Φ^+ and Φ^- of Φ such that $\Phi^+ * \mathbf{A} \notin \Gamma_\Sigma$ and $\Phi^- * \neg\mathbf{A} \notin \Gamma_\Sigma$ (since all finite subsets of Φ are in Γ_Σ). As $\Psi := \Phi^+ \cup \Phi^-$ is a finite subset of Φ, we have $\Psi \in \Gamma_\Sigma$. Furthermore, $\Psi * \mathbf{A} \in \Gamma_\Sigma$ or $\Psi * \neg\mathbf{A} \in \Gamma_\Sigma$, because Γ_Σ is saturated. Γ_Σ is closed under subsets, so $\Phi^+ * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi^- * \neg\mathbf{A} \in \Gamma_\Sigma$. This is a contradiction, so we can conclude that if $\Phi \in \Gamma_\Sigma$, then $\Phi * \mathbf{A} \in \Gamma'_\Sigma$ or $\Phi * \neg\mathbf{A} \in \Gamma'_\Sigma$. □

4.2 Hintikka Sets

Now, we define Hintikka sets, which are maximal elements in an abstract consistency class. Hintikka sets connect syntax with semantics, as they provide the basis for the model constructions in the model existence theorem 4.28.

Definition 4.13 (Σ-Hintikka Set). Let Γ_Σ be an abstract consistency class, then a set $\mathcal{H}$ is called a **Σ-Hintikka set for Γ_Σ**, iff it is maximal in Γ_Σ, i.e. iff for each sentence $\mathbf{D} \in cwff_o(\Sigma)$ such that $\mathcal{H} * \mathbf{D} \in \Gamma_\Sigma$, we already have $\mathbf{D} \in \mathcal{H}$.

In the following we discuss properties of Σ-Hintikka sets. Since we have different types of abstract consistency classes, depending on the additional requirements $\mathfrak{f}, \mathfrak{q}$ and $\mathfrak{b}$, we have to discuss different Hintikka lemmata.

Theorem 4.14 (Hintikka Lemma for $\mathfrak{Acc}_{\mathfrak{M}}$). *If Γ_Σ is a saturated $\mathfrak{Acc}_{\mathfrak{M}}$ and $\mathcal{H}$ is maximal in Γ_Σ, then the following statements hold for all $\mathbf{A}, \mathbf{B} \in cwff_o(\Sigma)$, $\mathbf{F} \in cwff_{\alpha\to o}(\Sigma)$ and $\mathbf{C}, \mathbf{D}, \mathbf{E} \in cwff_\alpha(\Sigma)$:*

$\overline{\nabla}_{c^a}$ $\mathbf{A} \notin \mathcal{H}$ *or* $\neg\mathbf{A} \notin \mathcal{H}$.

$\overline{\nabla}_{c^b}$ $\mathbf{A} \in \mathcal{H}$, *iff* $\neg\mathbf{A} \notin \mathcal{H}$.

$\overline{\nabla}_{c^c}$ $\neg\mathbf{A} \in \mathcal{H}$, *iff* $\mathbf{A} \notin \mathcal{H}$.

$\overline{\nabla}_\neg$ $(\neg\neg\mathbf{A}) \in \mathcal{H}$, *iff* $\mathbf{A} \in \mathcal{H}$.

$\overline{\nabla}_\beta$ *If* $\mathbf{A} \overset{*}{\leftrightarrow}_\beta \mathbf{B}$, *then* $\mathbf{A} \in \mathcal{H}$, *iff* $\mathbf{B} \in \mathcal{H}$.

$\overline{\nabla}_\vee$ $(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, *iff* $\mathbf{A} \in \mathcal{H}$ *or* $\mathbf{B} \in \mathcal{H}$.

$\overline{\nabla}_\wedge$ $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$, *iff* $\neg\mathbf{A} \in \mathcal{H}$ *and* $\neg\mathbf{B} \in \mathcal{H}$.

$\overline{\nabla}_\forall$ $\Pi^\alpha\mathbf{F} \in \mathcal{H}$, *iff for each* $\mathbf{D} \in cwff_\alpha(\Sigma)$ *we have* $\mathbf{F}\mathbf{D} \in \mathcal{H}$.

$\overline{\nabla}_\exists$ $\neg\Pi^\alpha\mathbf{F} \in \mathcal{H}$, *iff there is a* $\mathbf{D} \in cwff_\alpha(\Sigma)$ *such that* $\neg\mathbf{F}\mathbf{D} \in \mathcal{H}$.

$\overline{\nabla}_{\doteq^r}$ $\mathbf{A} \doteq^\alpha \mathbf{A} \in \mathcal{H}$

$\overline{\nabla}_{\doteq^c}$ *If* $\mathbf{F}[\mathbf{C}]_p \in \mathcal{H}$ *and* $\mathbf{C} \doteq^\alpha \mathbf{D} \in \mathcal{H}$, *then* $\mathbf{F}[\mathbf{D}]_p \in \mathcal{H}$

$\overline{\nabla}_{\doteq^{sy}}$ $\mathbf{C} \doteq^\alpha \mathbf{D} \in \mathcal{H}$, *iff* $\mathbf{D} \doteq^\alpha \mathbf{C} \in \mathcal{H}$

$\overline{\nabla}_{\doteq^{tr}}$ $\mathbf{C} \doteq^\alpha \mathbf{D} \in \mathcal{H}$ *and* $\mathbf{D} \doteq^\alpha \mathbf{E} \in \mathcal{H}$, *then* $\mathbf{C} \doteq^\alpha \mathbf{E} \in \mathcal{H}$

$\overline{\nabla}_t$ $(\mathbf{A} \vee \neg\mathbf{A}) \in \mathcal{H}$ *for any sentence* $\mathbf{A}$.

Proof:

$\overline{\nabla}_{c^a}$ By 4.8.

$\overline{\nabla}_{c^b}, \overline{\nabla}_{c^c}$ Both are direct consequences of the saturation of Γ_Σ and $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_\neg$ If $\neg\neg\mathbf{A} \in \mathcal{H}$, then $\mathcal{H} * \mathbf{A} \in \Gamma_\Sigma$ by $\nabla_\neg$. The maximality of $\mathcal{H}$ now gives us that $\mathbf{A} \in \mathcal{H}$. To obtain the converse, let us assume that $\mathbf{A} \in \mathcal{H}$. Then by $\overline{\nabla}_{c^b}$ we know that $\neg\mathbf{A} \notin \mathcal{H}$ and by $\overline{\nabla}_{c^c}$ $\neg\neg\mathbf{A} \in \mathcal{H}$.

$\overline{\nabla}_\beta$ Suppose $\mathbf{A} \overset{*}{\leftrightarrow}_\beta \mathbf{B}$. Since β-reduction is terminating and confluent there is unique $\mathbf{C}$ such that $\mathbf{C}$ is the β-normal-form of $\mathbf{A}$ and $\mathbf{B}$. Without loss of generality we show that if $\mathbf{A} \in \mathcal{H}$, then $\mathbf{B} \in \mathcal{H}$. For that we suppose that $\mathbf{A} \in \mathcal{H}$ but $\mathbf{B} \notin \mathcal{H}$. From the latter we get by by $\overline{\nabla}_{c^c}$ that $\neg\mathbf{B} \in \mathcal{H}$. Note that the β-normal-form of $\mathbf{A}$ is $\mathbf{C}$ and of $\neg\mathbf{B}$ is $\neg\mathbf{C}$. By ∇_β and the maximality of $\mathcal{H}$ we know that $\{\mathbf{C}, \neg\mathbf{C}\} \in \mathcal{H}$ which contradicts $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_\vee$ We get the first direction by $\nabla_\vee$ and the the maximality of $\mathcal{H}$. For the converse direction let us assume that $\mathbf{A} \in \mathcal{H}$ or $\mathbf{B} \in \mathcal{H}$ but $(\mathbf{A} \vee \mathbf{B}) \notin \mathcal{H}$. Then by $\overline{\nabla}_{c^c}$ we get $\neg(\mathbf{A} \vee \mathbf{B}) \in \mathcal{H}$ and by the first direction of $\overline{\nabla}_\wedge$ we have $\{\neg\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ which contradicts the assumption with $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_\wedge$ Analogous to the $\overline{\nabla}_\vee$ case; Note that the argumentation is not circular. In both cases we use the forward direction of the counterpart to verify the backward direction, whereas forward directions are proven directly. The same holds for the proofs of $\overline{\nabla}_\forall$ and $\overline{\nabla}_\exists$ below.

$\overline{\nabla}_\forall$ Again, we get the first direction by $\nabla_\forall$ and the maximality of $\mathcal{H}$. For the converse direction let us assume that $\mathbf{FD} \in \mathcal{H}$ for each $\mathbf{D} \in cwff_\alpha(\Sigma)$, but $\Pi^\alpha\mathbf{F} \notin \mathcal{H}$. Then by $\overline{\nabla}_{c^c}$ $\neg\Pi^\alpha\mathbf{F} \in \mathcal{H}$ and by the first direction of $\overline{\nabla}_\exists$ there is a $\mathbf{D} \in cwff_\alpha(\Sigma)$, such that $\neg\mathbf{FD} \in \mathcal{H}$ which is a contradiction.

$\overline{\nabla}_\exists$ Analogous to $\overline{\nabla}_\forall$.

$\overline{\nabla}_{\doteq^r}$ Suppose $\mathbf{A} \doteq^\alpha \mathbf{A} \notin \mathcal{H}$. By $\overline{\nabla}_{c^b}$, the definition of $\doteq$, $\overline{\nabla}_\exists$ and $\overline{\nabla}_\mathfrak{b}$ we have $\neg(\neg\mathbf{QA} \vee \mathbf{QA})) \in \mathcal{H}$ for a $\mathbf{Q} \in cwff_{\beta\to o}(\Sigma)$. Applying $\overline{\nabla}_\wedge$ contradicts $\overline{\nabla}_{c^b}$.

$\overline{\nabla}_{\doteq^c}$ Suppose $\mathbf{F}[\mathbf{C}]_p \in \mathcal{H}$ and $\mathbf{C} \doteq \mathbf{D} \in \mathcal{H}$. From the latter we obtain $(\lambda P.\neg P\mathbf{C} \vee P\mathbf{D})(\lambda X.\mathbf{F}[X]_p) \in \mathcal{H}$ by the definition of $\doteq$ and $\overline{\nabla}_\forall$. Note that X is free for $\mathbf{F}[Y]_p$ so we have $\neg\mathbf{F}[\mathbf{C}]_p \vee \mathbf{F}[\mathbf{D}]_p \in \mathcal{H}$ by $\overline{\nabla}_\beta$. From this we conclude with $\overline{\nabla}_\vee$ that $\neg\mathbf{F}[\mathbf{C}]_p \in \mathcal{H}$ or $\mathbf{F}[\mathbf{D}]_p \in \mathcal{H}$. Since the first option contradicts our assumption with $\overline{\nabla}_{c^a}$, it must be the case that $\mathbf{F}[\mathbf{B}]_p \in \mathcal{H}$.

$\overline{\nabla}_{\doteq^{sy}}$ By $\overline{\nabla}_{\doteq^r}$ and $\overline{\nabla}_{\doteq^c}$.

$\overline{\nabla}_{\doteq^{tr}}$ By $\overline{\nabla}_{\doteq^r}$, $\overline{\nabla}_{\doteq^c}$ and $\overline{\nabla}_{\doteq^{sy}}$.

$\overline{\nabla}_t$ Saturation of Γ_Σ and maximality of $\mathcal{H}$ entails that $\mathbf{A} \in \mathcal{H}$ or $\neg\mathbf{A} \in \mathcal{H}$. We now get the assertion by $\overline{\nabla}_\vee$. □

Depending on the kind of abstract consistency class we are considering, Hintikka sets have different properties. We discuss this different properties in the Hintikka lemmata below.

Theorem 4.15 (Hintikka Lemma for $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{f}}$). *If Γ_Σ is a saturated $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{f}}$ and $\mathcal{H}$ is maximal in Γ_Σ, then for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in cwff_o(\Sigma)$*

$\overline{\nabla}_\mathfrak{f}$ *If $\mathbf{A} \overset{*}{\leftrightarrow}_{\beta\eta} \mathbf{B}$, then $\mathbf{A} \in \mathcal{H}$ iff $\mathbf{B} \in \mathcal{H}$.*

Proof: Analogous to $\overline{\nabla}_\beta$ in lemma 4.14 □

Theorem 4.16 (Hintikka Lemma for $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{q}}$). *If Γ_Σ is a saturated $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{q}}$ and $\mathcal{H}$ is maximal in Γ_Σ, then for all $\mathbf{C} \in cwff_\alpha(\Sigma)$, $\mathbf{F}, \mathbf{G} \in cwff_{\alpha\to\beta}(\Sigma)$:*

$\overline{\nabla}_\mathfrak{q}$ *$\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \mathcal{H}$, iff there is a $\mathbf{C} \in cwff_\alpha(\Sigma)$, such that $\neg(\mathbf{FC} \doteq^\beta \mathbf{GC}) \in \mathcal{H}$.*

$\overline{\nabla}_{\mathfrak{q}'}$ *$\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G} \in \mathcal{H}$, iff $\mathbf{FC} \doteq^\beta \mathbf{GC} \in \mathcal{H}$ for all $\mathbf{C} \in cwff_\alpha(\Sigma)$*

Proof:

$\overline{\nabla}_\mathfrak{q}$ We get the first direction by the definition of $\doteq$, $\nabla_\mathfrak{q}$ and the maximality of $\mathcal{H}$. For the converse let us suppose that $\neg(\mathbf{FC} \doteq \mathbf{GC}) \in \mathcal{H}$ but $\neg(\mathbf{F} \doteq \mathbf{G}) \notin \mathcal{H}$. From the latter we know by $\overline{\nabla}_{c^b}$, that $\mathbf{F} \doteq \mathbf{G} \in \mathcal{H}$ and by $\overline{\nabla}_{\doteq^c}$ we have that $\neg(\mathbf{GC} \doteq \mathbf{GC}) \in \mathcal{H}$ which contradicts $\overline{\nabla}_{\doteq^r}$ and $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_{\mathfrak{q}'}$ Suppose $\mathbf{F} \doteq \mathbf{G} \in \mathcal{H}$ but $\mathbf{FC} \doteq \mathbf{GC} \notin \mathcal{H}$ which means by $\overline{\nabla}_{c^c}$, that $\neg(\mathbf{FC} \doteq \mathbf{GC}) \in \mathcal{H}$. From this we get by the definition of $\doteq$, $\overline{\nabla}_\exists$ and $\overline{\nabla}_\beta$, that $\neg(\neg\mathbf{Q}(\mathbf{FC}) \vee \mathbf{Q}(\mathbf{GC})) \in \mathcal{H}$ for some $\mathbf{Q} \in wff_{\alpha\to o}(\Sigma)$. On the other hand we know from $\mathbf{F} \doteq \mathbf{G} \in \mathcal{H}$ by the definition of $\doteq$ and $\overline{\nabla}_\forall$ that $(\lambda P_{(\alpha\to\beta)\to o}.\neg P\mathbf{F} \vee P\mathbf{G})(\lambda X_{\alpha\to\beta}.\mathbf{Q}(X\mathbf{C})) \in \mathcal{H}$, and hence by $\overline{\nabla}_\beta$ that $\neg\mathbf{Q}(\mathbf{FC}) \vee \mathbf{Q}(\mathbf{GC}) \in \mathcal{H}$ which contradicts $\overline{\nabla}_{c^a}$. For the converse assume that

$\mathbf{FC} \doteq \mathbf{GC} \in \mathcal{H}$ for all $\mathbf{C} \in \mathcal{H}$ but $\mathbf{F} \doteq \mathbf{G} \notin \mathcal{H}$. We get by $\overline{\nabla}_{c^e}$ that $\neg(\mathbf{F} \doteq \mathbf{G}) \in \mathcal{H}$ which contradicts the assumption with $\overline{\nabla}_{\mathfrak{q}}$ and $\overline{\nabla}_{c^a}$. □

Theorem 4.17 (Hintikka Lemma for $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{b}}$). *If Γ_Σ is a saturated $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{b}}$ and $\mathcal{H}$ is maximal in Γ_Σ, then for all $\mathbf{A}, \mathbf{B} \in wff_o(\Sigma)$:*

$\overline{\nabla}_\mathfrak{b}$ $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$, *iff* $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ *or* $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$

$\overline{\nabla}_{\mathfrak{b}'}$ $(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$, *iff* $\{\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ *or* $\{\neg\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$.

$\overline{\nabla}_{\doteq i}$ $(\mathbf{A} \Leftrightarrow \mathbf{B}) \in \mathcal{H}$, *iff* $(\mathbf{A} \doteq^o \mathbf{B}) \in \mathcal{H}$.

$\overline{\nabla}_{\doteq tc}$ *Either* $\mathbf{A} \doteq^o \mathbf{B} \in \mathcal{H}$ *or* $\mathbf{A} \doteq^o \neg\mathbf{B} \in \mathcal{H}$.

$\overline{\nabla}_{\doteq tf}$ $\neg(\mathbf{T}_o \doteq \mathbf{F}_o) \in \mathcal{H}$, *if* $\mathbf{T}_o$ *and* $\mathbf{F}_o$ *are defined as in lemma 3.25.*

$\overline{\nabla}_{\doteq tf'}$ *Either* $\mathbf{A} \doteq^o \mathbf{T}_o \in \mathcal{H}$ *or* $\mathbf{A} \doteq^o \mathbf{F}_o \in \mathcal{H}$.

Proof:

$\overline{\nabla}_\mathfrak{b}$ We get the first direction by the definition of $\doteq$, $\nabla_\mathfrak{b}$ and the maximality of $\mathcal{H}$. Now assume that $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ or $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ but $\neg(\mathbf{A} \doteq \mathbf{B}) \notin \mathcal{H}$. From the latter we know by the definition of $\doteq$ and $\overline{\nabla}_\forall$ that $\{(\lambda P_{o\to o}.\neg P\mathbf{A} \vee P\mathbf{B})(\lambda X_o.X), (\lambda P_{o\to o}.\neg P\mathbf{A} \vee P\mathbf{B})(\lambda X_o.\neg X)\} \subseteq \mathcal{H}$ and by $\overline{\nabla}_\beta$ and $\overline{\nabla}_\vee$ that one of $\{\neg\mathbf{A}, \neg\neg\mathbf{A}\}$, $\{\mathbf{B}, \neg\mathbf{B}\}$, $\{\neg\mathbf{A}, \neg\mathbf{B}\}$ or $\{\mathbf{B}, \neg\neg\mathbf{A}\}$ must be a subset of $\mathcal{H}$. All four cases contradict $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_{\mathfrak{b}'}$ Since Γ_Σ is saturated we have $\mathbf{A} \in \mathcal{H}$ or $\neg\mathbf{A} \in \mathcal{H}$. From this we easily get the first direction by $\overline{\nabla}_{\doteq c}$. For the converse suppose that $\{\mathbf{A}, \mathbf{B}\} \in \mathcal{H}$ or $\{\neg\mathbf{A}, \neg\mathbf{B}\} \in \mathcal{H}$ but $(\mathbf{A} \doteq \mathbf{B}) \notin \mathcal{H}$ which means by $\overline{\nabla}_{c^e}$ that $\neg(\mathbf{A} \doteq \mathbf{B}) \in \mathcal{H}$. By $\overline{\nabla}_\mathfrak{b}$ we have $\{\neg\mathbf{A}, \mathbf{B}\} \in \mathcal{H}$ or $\{\mathbf{A}, \neg\mathbf{B}\} \in \mathcal{H}$. In each of the four cases the contradiction follows by $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_{\doteq i}$ If we assume $(\mathbf{A} \Leftrightarrow \mathbf{B}) \in \mathcal{H}$, then by the definition of $\Leftrightarrow$ and $\overline{\nabla}_\wedge$ we have $\{\neg\mathbf{A} \vee \mathbf{B}, \mathbf{A} \vee \neg\mathbf{B}\} \subseteq \mathcal{H}$, and by $\overline{\nabla}_\vee$ that $\{\neg\mathbf{A}, \mathbf{A}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{B}, \mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ or $\{\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$. Note that the first two alternatives are impossible because of $\overline{\nabla}_{c^a}$. Now we assume that $\mathbf{A} \doteq \mathbf{B} \notin \mathcal{H}$ from which we obtain by $\overline{\nabla}_{c^e}$ and $\overline{\nabla}_\mathfrak{b}$ that $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ or $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$. We have to consider four cases and in each case we get a contradiction with $\overline{\nabla}_{c^a}$.

$\overline{\nabla}_{\doteq tc}$ Assume that $\mathbf{A} \doteq \mathbf{B} \notin \mathcal{H}$ and $\mathbf{A} \doteq \neg\mathbf{B} \notin \mathcal{H}$. By $\overline{\nabla}_{c^e}$ we have $\neg(\mathbf{A} \doteq \mathbf{B}) \in \mathcal{H}$ and $\neg(\mathbf{A} \doteq \neg\mathbf{B}) \in \mathcal{H}$, and by $\overline{\nabla}_\mathfrak{b}$ we get from the former that $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ or $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ and from the latter that $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$ or $\{\neg\mathbf{A}, \neg\neg\mathbf{B}\} \subseteq \mathcal{H}$. We have to consider four cases and in each we get a contradiction with $\overline{\nabla}_{c^a}$. Analogous we can show with $\overline{\nabla}_{\mathfrak{b}'}$ that $\mathbf{A} \doteq \mathbf{B} \in \mathcal{H}$ and $\mathbf{A} \doteq \neg\mathbf{B} \in \mathcal{H}$ leads to a contradiction.

$\overline{\nabla}_{\doteq tf}$ From $\overline{\nabla}_t$ we know that $\mathbf{T}_o \in \mathcal{H}$. Hence by $\overline{\nabla}_{c^b}$ and $\overline{\nabla}_{c^e}$ that $\neg\mathbf{F}_o \in \mathcal{H}$ and finally by $\overline{\nabla}_\mathfrak{b}$ we get $\neg(\mathbf{T}_o \doteq^o \mathbf{F}_o) \in \mathcal{H}$.

$\overline{\nabla}_{\doteq tf'}$ Follows immediately from $\overline{\nabla}_{\doteq tc}$. □

4.3 Primitive Equality

Next we will introduce abstract consistency properties for primitive equality. We have different options, e.g. we could introduce primitive equality by postulating $=$ to be a functional congruence relation or alternatively we could state properties connecting $=$ with $\doteq$.

Our concrete choice, namely a property postulating reflexivity and substitutivity of $=$, is motivated from a practical point of view, as we believe that reflexivity and substitutivity are more easy to verify in practical applications.

Definition 4.18 (Abstract Consistency with Primitive Equality). Let Σ be signature and let Γ_Σ be a $\mathfrak{Acc}_\mathfrak{M}$, then we define the following condition, where $\Phi \in \Gamma_\Sigma$:

$\nabla_\mathfrak{e}$ (r) $\neg(\mathbf{A} =^\alpha \mathbf{A}) \notin \Phi$
(s) if $\mathbf{F}[\mathbf{A}]_p \in \Phi$ and $\mathbf{A} = \mathbf{B} \in \Phi$, then $\Phi * \mathbf{F}[\mathbf{B}]_p \in \Gamma_\Sigma$

Using this properties we introduce the following abstract consistency classes $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{e}}$ and $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{eb}}}$ based upon the definition of an $\mathfrak{Acc}_\mathfrak{M}$.

Remark 4.19. Just as in the case with Leibniz equality, we can extend a abstract consistency class with primitive equality so that it is compact.

Proof: We proceed just as in the proof of Lemma 4.12 but check the cases for $\nabla_{\!\mathfrak{e}}(\mathrm{r})$ and $\nabla_{\!\mathfrak{e}}(\mathrm{s})$.

For $\nabla_{\!\mathfrak{e}}(\mathrm{r})$ let $\Phi \in \Gamma'_{\!\Sigma}$ and suppose there is an $\mathbf{A} \in \mathit{wff}_o(\Sigma)$ with $\neg(\mathbf{A} = \mathbf{A}) \in \Phi$. Then $\{\neg(\mathbf{A} = \mathbf{A})\} \in \Gamma_{\!\Sigma}$ contradicting $\nabla_{\!\mathfrak{e}}(\mathrm{r})$.

For $\nabla_{\!\mathfrak{e}}(\mathrm{s})$ Let $\Phi \in \Gamma'_{\!\Sigma}$, $\{\mathbf{F}[\mathbf{A}]_p, \mathbf{A} = \mathbf{B}\} \subseteq \Phi$, Ψ be any finite subset of $\Phi * \mathbf{F}[\mathbf{B}]_p$ and $\Theta := (\Psi \setminus \{\mathbf{F}[\mathbf{B}]_p\}) \cup \{\mathbf{F}[\mathbf{A}]_p, \mathbf{A} = \mathbf{B}\}$. Θ is a finite subset of Φ, so $\Theta \in \Gamma_{\!\Sigma}$. Since $\Gamma_{\!\Sigma}$ is an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{eb}}}$ and $\{\mathbf{F}[\mathbf{A}]_p, \mathbf{A} = \mathbf{B}\} \subseteq \Theta$, we get $\Theta * \mathbf{F}[\mathbf{B}]_p \in \Gamma_{\!\Sigma}$ by $\nabla_{\!\mathfrak{e}}(\mathrm{s})$. We know that $\Psi \subseteq \Theta * \mathbf{F}[\mathbf{B}]_p$ and $\Gamma_{\!\Sigma}$ is closed under subsets, so $\Psi \in \Gamma_{\!\Sigma}$. Thus every finite subset Ψ of $\Phi * \mathbf{F}[\mathbf{B}]_p$ is in $\Gamma_{\!\Sigma}$ and therefore by definition $\Phi * \mathbf{F}[\mathbf{B}]_p \in \Gamma'_{\!\Sigma}$. □

The next lemma discusses the connection between Leibniz equality and primitive equality in case we are considering an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{e}}}$.

Lemma 4.20 (Leibniz vs. Primitive Equality). *Let $\Gamma_{\!\Sigma}$ be a saturated $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{e}}}$. For all $\Phi \in \Gamma_{\!\Sigma}$, all $\mathbf{A}, \mathbf{B} \in \mathit{wff}_o(\Sigma)$ and $\mathbf{F}, \mathbf{G} \in \mathit{wff}_{\alpha\to\beta}(\Sigma)$ holds:*

1. *If $\neg(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \Phi$ then $\Phi * \neg(\mathbf{A} =^\alpha \mathbf{B}) \in \Gamma_{\!\Sigma}$*
2. *If $\neg(\mathbf{A} =^\alpha \mathbf{B}) \in \Phi$ then $\Phi * \neg(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \Gamma_{\!\Sigma}$*
3. *If $\mathbf{A} \doteq^\alpha \mathbf{B} \in \Phi$ then $\Phi * \mathbf{A} =^\alpha \mathbf{B} \in \Gamma_{\!\Sigma}$*
4. *If $\mathbf{A} =^\alpha \mathbf{B} \in \Phi$ then $\Phi * \mathbf{A} \doteq^\alpha \mathbf{B} \in \Gamma_{\!\Sigma}$*
5. *If $\neg(\mathbf{F} =^{\alpha\to\beta} \mathbf{G}) \in \Phi$ then $\Phi * \neg(\mathbf{F}w =^\beta \mathbf{G}w) \in \Gamma_{\!\Sigma}$ for any constant $w \in \Sigma_\alpha$, which does not occur in Φ.*

Proof:

1. Suppose $\neg(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \Phi$ but $\Phi * \neg(\mathbf{A} =^\alpha \mathbf{B}) \notin \Gamma_{\!\Sigma}$. Since $\Gamma_{\!\Sigma}$ is saturated we have $\Phi * \mathbf{A} =^\alpha \mathbf{B} \in \Gamma_{\!\Sigma}$ and by $\nabla_{\!\mathfrak{e}}(\mathrm{s})$, that $\Phi * \mathbf{A} =^\alpha \mathbf{B} * \neg(\mathbf{B} \doteq^\alpha \mathbf{B}) \in \Gamma_{\!\Sigma}$. From the definition of $\doteq$ we further conclude with $\nabla_{\!\exists}$ that $\Phi * \mathbf{A} =^\alpha \mathbf{B} * \neg(\mathbf{B} \doteq^\alpha \mathbf{B}) * \neg(\neg p\mathbf{B} \vee p\mathbf{B}) \in \Gamma_{\!\Sigma}$ for any constant $p \in \Sigma_{\alpha\to o}$. From this we get the contradiction with $\nabla_{\!\wedge}$ and lemma 4.8.
2. Suppose $\neg(\mathbf{A} =^\alpha \mathbf{B}) \in \Phi$ but $\Phi * \neg(\mathbf{A} \doteq^\alpha \mathbf{B}) \notin \Gamma_{\!\Sigma}$. Since $\Gamma_{\!\Sigma}$ is saturated we have $\Phi * \mathbf{A} \doteq^\alpha \mathbf{B} \in \Gamma_{\!\Sigma}$ and by definition of $\doteq$, $\nabla_{\!\forall}$ and the subset closure of $\Gamma_{\!\Sigma}$ that $\Phi * (\lambda P_{\alpha\to o}.\neg P\mathbf{A} \vee P\mathbf{B})(\lambda X_\alpha.\mathbf{A} = X) \in \Gamma_{\!\Sigma}$. By $\nabla_{\!\beta}$, $\nabla_{\!\vee}$ and the subset closure of $\Gamma_{\!\Sigma}$ we finally get that $\Phi * \neg(\mathbf{A} = \mathbf{A}) \in \Gamma_{\!\Sigma}$ or $\Phi * \mathbf{A} = \mathbf{B} \in \Gamma_{\!\Sigma}$. The former is contradictory with $\nabla_{\!\mathfrak{e}}(\mathrm{r})$ and lemma 4.8, and the latter with the assumption $\neg(\mathbf{A} =^\alpha \mathbf{B}) \in \Phi$ and lemma 4.8.
3. Suppose $\mathbf{A} \doteq^\alpha \mathbf{B} \in \Phi$ but $\Phi * \mathbf{A} =^\alpha \mathbf{B} \notin \Gamma_{\!\Sigma}$. Since $\Gamma_{\!\Sigma}$ is saturated we have $\Phi * \neg(\mathbf{A} =^\alpha \mathbf{B}) \in \Gamma_{\!\Sigma}$ and by (2) and the subset closure of $\Gamma_{\!\Sigma}$ that $\Phi * \neg(\mathbf{A} \doteq^\alpha \mathbf{B}) \in \Gamma_{\!\Sigma}$ which contradicts the assumption with lemma 4.8.
4. Analogous to (3) with (1).
5. From $\neg(\mathbf{F} =^{\alpha\to\beta} \mathbf{B}) \in \Phi$ we can derive with (2), $\nabla_{\!\mathfrak{q}}$, (1) and the subset closure of $\Gamma_{\!\Sigma}$ that $\Phi * \neg(\mathbf{FC} =^{\alpha\to\beta} \mathbf{BC}) \in \Gamma_{\!\Sigma}$. □

Remark 4.21. Lemma 4.20 shows that in an $\mathfrak{M}_{\mathfrak{e}}$ the symbol $=$ defines the same relation as $\doteq$, namely a functional congruence relation modulo v. And if we are considering an $\mathfrak{M}_{\mathfrak{eb}}$ then both describe the equality relation. This shows that the conditions $\nabla_{\!\mathfrak{e}}(\mathrm{r})$ and $\nabla_{\!\mathfrak{e}}(\mathrm{s})$ are sufficient for this purpose. We could alternatively introduce primitive equality by requiring the statements 1. and 2. of lemma 4.20, but this would lead to more complicated proof obligations when proving the completeness of calculi with primitive equality.

We now discuss two new Hintikka lemmata, which take the logical nature of $=$ into account

Theorem 4.22 (Hintikka Lemma for $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{e}}}$). *If $\Gamma_{\!\Sigma}$ is a saturated $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{e}}}$ and $\mathcal{H}$ is maximal in $\Gamma_{\!\Sigma}$, then the following statements hold for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathit{wff}_\alpha(\Sigma)$, $\mathbf{F}, \mathbf{G} \in \mathit{wff}_{\alpha\to\beta}(\Sigma)$ and $\mathbf{D}, \mathbf{E} \in \mathit{wff}_o(\Sigma)$:*

$\overline{\nabla}_{\!=^r}$ $(\mathbf{A} =^\alpha \mathbf{A}) \in \mathcal{H}$.

$\overline{\nabla}_{\!=^c}$ *If $\mathbf{D}[\mathbf{A}]_p \in \mathcal{H}$ and $\mathbf{A} =^\alpha \mathbf{B} \in \mathcal{H}$, then $\mathbf{D}[\mathbf{B}]_p \in \mathcal{H}$.*

$\overline{\nabla}_{\!=^{sy}}$ $\mathbf{A} =^\alpha \mathbf{B} \in \mathcal{H}$, *iff* $(\mathbf{B} =^\alpha \mathbf{A}) \in \mathcal{H}$.

$\overline{\nabla}_{=^{tr}}$ $\mathbf{A} =^{\alpha} \mathbf{B} \in \mathcal{H}$ *and* $\mathbf{B} =^{\alpha} \mathbf{C} \in \mathcal{H}$*, then* $\mathbf{A} =^{\alpha} \mathbf{C} \in \mathcal{H}$

$\overline{\nabla}_{=^{q}}$ $\neg(\mathbf{F} =^{\alpha\to\beta} \mathbf{G}) \in \mathcal{H}$*, iff there is a* $\mathbf{C} \in wff_{\alpha}(\Sigma)$*, such that* $\neg(\mathbf{FC} =^{\beta} \mathbf{GC}) \in \mathcal{H}$.

$\overline{\nabla}_{=^{q'}}$ $\mathbf{F} =^{\alpha\to\beta} \mathbf{G} \in \mathcal{H}$*, iff* $\mathbf{FC} =^{\beta} \mathbf{GC} \in \mathcal{H}$ *for all* $\mathbf{C} \in wff_{\alpha}(\Sigma)$.

$\overline{\nabla}_{=^{\doteq}}$ $\mathbf{A} \doteq^{\alpha} \mathbf{B} \in \mathcal{H}$*, iff* $\mathbf{A} =^{\alpha} \mathbf{B} \in \mathcal{H}$.

$\overline{\nabla}_{=^{\doteq'}}$ $\neg(\mathbf{A} \doteq^{\alpha} \mathbf{B}) \in \mathcal{H}$*, iff* $\neg(\mathbf{A} =^{\alpha} \mathbf{B}) \in \mathcal{H}$.

Proof:

$\overline{\nabla}_{=^{r}}$ Follows by $\nabla_{\mathfrak{e}}(\mathrm{r})$ and $\overline{\nabla}_{c^b}$.

$\overline{\nabla}_{=^{c}}$ By maximality of $\mathcal{H}$ and $\nabla_{\mathfrak{e}}(\mathrm{s})$.

$\overline{\nabla}_{=^{sy}}, \overline{\nabla}_{=^{tr}}$ By $\overline{\nabla}_{=^{r}}$ and $\overline{\nabla}_{=^{c}}$

$\overline{\nabla}_{=^{\doteq}}$ By maximality of $\mathcal{H}$, 4.20(3.) and 4.20(4.)

$\overline{\nabla}_{=^{\doteq'}}$ By maximality of $\mathcal{H}$, 4.20(1.) and 4.20(2.)

$\overline{\nabla}_{=^{q}}$ Follows from $\overline{\nabla}_{\mathfrak{q}}$ with $\overline{\nabla}_{=^{\doteq}}$.

$\overline{\nabla}_{=^{q'}}$ Follows from $\overline{\nabla}_{\mathfrak{q}}$ with $\overline{\nabla}_{=^{\doteq'}}$. □

Theorem 4.23 (Hintikka Lemma for $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{eb}}}$). *If* Γ_{Σ} *is a saturated* $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{eb}}}$ *and* $\mathcal{H}$ *is maximal in* Γ_{Σ}*, then for all* $\mathbf{A}, \mathbf{B} \in wff_{o}(\Sigma)$*:*

$\overline{\nabla}_{=^{b}}$ $\neg(\mathbf{A} =^{o} \mathbf{B}) \in \mathcal{H}$*, iff* $\{\neg\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ *or* $\{\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$.

$\overline{\nabla}_{=^{b'}}$ $\mathbf{A} =^{o} \mathbf{B} \in \mathcal{H}$*, iff* $\{\mathbf{A}, \mathbf{B}\} \subseteq \mathcal{H}$ *or* $\{\neg\mathbf{A}, \neg\mathbf{B}\} \subseteq \mathcal{H}$.

$\overline{\nabla}_{=^{i}}$ $\mathbf{A} \Leftrightarrow \mathbf{B} \in \mathcal{H}$*, iff* $\mathbf{A} = \mathbf{B} \in \mathcal{H}$.

$\overline{\nabla}_{=^{tc}}$ *Either* $\mathbf{A} = \mathbf{B} \in \mathcal{H}$ *or* $\mathbf{A} = \neg\mathbf{B} \in \mathcal{H}$.

Proof: The statements follow direct from their counterparts $\overline{\nabla}_{\mathfrak{b}}$ – $\overline{\nabla}_{\doteq^{tc}}$ in lemma 4.17 with the help of $\overline{\nabla}_{=^{\doteq}}$ and $\overline{\nabla}_{=^{\doteq'}}$. □

4.4 Model Existence

Next we come to the proof of the abstract extension lemma, which will nearly immediately yield the model existence theorems. For the proof we adapt the construction of Henkin's completeness proof from [Hen50].

Theorem 4.24 (Abstract Extension Lemma). *Let* Σ *be a signature,* Γ_{Σ} *be a compact abstract consistency class and let* $H \in \Gamma_{\Sigma}$ *be a sufficiently* Σ*-pure set of* Σ*-sentences. Then there exists a* Σ*-Hintikka set* $\mathcal{H}$ *for* Γ_{Σ}*, such that* $H \subseteq \mathcal{H}$.

Proof: We construct $\mathcal{H}$ by inductively constructing a sequence of sets $\mathcal{H}^i$ such that $\mathcal{H}^i \in \Gamma_{\Sigma}$. Then the Σ-Hintikka set is $\mathcal{H} := \bigcup_{i\in\mathbb{N}} \mathcal{H}^i \in \Gamma_{\Sigma}$.
Let $\mathbf{A}_1, \mathbf{A}_2, \ldots$ be a sequence of all sentences in $wff_{o}(\Sigma)$. We define $\mathcal{H}^0 := H$ and the set $\mathcal{H}^{n+1}$ according to the table 3. Since the construction is uniform for all kinds of abstract consistency classes $\mathcal{H}^{n+1}$ depends on the respective kind of abstract consistency class Γ_{Σ} we are interested in and in the properties of $\mathbf{A}_n$ with respect to this Γ_{Σ}.

Next we show by induction, that $\mathcal{H}^n \in \Gamma_{\Sigma}$ for all $n \in \mathbb{N}$. The base case holds by construction (for all kinds of abstract consistency classes). So let $\mathcal{H}^n * \in \Gamma_{\Sigma}$. We have to show that $\mathcal{H}^{n+1} \in \Gamma_{\Sigma}$. This is trivial in case $\mathcal{H}^n * \mathbf{A}_n \notin \Gamma_{\Sigma}$ (again for all abstract consistency classes). In case $\mathcal{H}^n * \mathbf{A}_n \in \Gamma_{\Sigma}$ we have to consider four sub cases:

1. If $\mathbf{A}_n$ is of form $\neg\Pi^{\alpha}\mathbf{B}$, then we get the conclusion trivially by $\nabla_{\exists}$ (for all cases).
2. If $\mathbf{A}_n$ is of form $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$ the conclusion is either trivial (by $\nabla_{\exists}$ in case of an $\mathfrak{Acc}_{\mathfrak{M}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{b}}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{f}}}$ or $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{fb}}}$) or follows by $\nabla_{\mathfrak{q}}$.
3. If $\mathbf{A}_n$ is of form $\neg(\mathbf{F} =^{\alpha\to\beta} \mathbf{G})$ the conclusion is either trivial (by $\nabla_{\exists}$ in case of an $\mathfrak{Acc}_{\mathfrak{M}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{b}}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{f}}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{fb}}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{q}}}$, $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}$) or follows by 4.20(5).
4. If $\mathbf{A}_n$ is of any other form, then the conclusion is trivial (for all cases).

$\mathcal{H}^{n+1}$		$\mathfrak{Acc}_{\mathfrak{M}}/\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{b}}}/$ $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{f}}}/\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{fb}}}$	$\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{q}}}/\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}/$	$\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{e}}}/\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{eb}}}/$
$\mathcal{H}^n * \mathbf{A}_n \notin \Gamma_\Sigma$		$\mathcal{H}^n$	$\mathcal{H}^n$	$\mathcal{H}^n$
$\mathcal{H}^n * \mathbf{A}_n \in \Gamma_\Sigma$ and	$\mathbf{A}_n$ of form $\neg\Pi^\alpha\mathbf{B}$	$\mathcal{H}^n * \mathbf{A}_n * \neg\mathbf{B}w$	$\mathcal{H}^n * \mathbf{A}_n * \neg\mathbf{B}w$	$\mathcal{H}^n * \mathbf{A}_n * \neg\mathbf{B}w$
	$\mathbf{A}_n$ of form $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$	$\mathcal{H}^n * \mathbf{A}_n$	$\mathcal{H}^n * \mathbf{A}_n * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w)$	$\mathcal{H}^n * \mathbf{A}_n * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w)$
	$\mathbf{A}_n$ of form $\neg(\mathbf{F} =^{\alpha\to\beta} \mathbf{G})$	$\mathcal{H}^n * \mathbf{A}_n$	$\mathcal{H}^n * \mathbf{A}_n$	$\mathcal{H}^n * \mathbf{A}_n * \neg(\mathbf{F}w =^\beta \mathbf{G}w)$
	$\mathbf{A}_n$ of other form	$\mathcal{H}^n * \mathbf{A}_n$	$\mathcal{H}^n * \mathbf{A}_n$	$\mathcal{H}^n * \mathbf{A}_n$
$w \in \Sigma_\alpha$ is a constant which is fresh for $\mathcal{H}^n$				

Figure 3: Construction of $\mathcal{H}^{n+1}$. *How to read the table*: Assume Γ_Σ is an an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{q}}}$ and $\mathbf{A}_n$ is of form $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G})$. The table defines $\mathcal{H}^{n+1}$ to be $\mathcal{H}^n * \mathbf{A}_n * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w)$ for a fresh $w \in \Sigma_\alpha$ in case $\mathbf{A}_n \in \Gamma_\Sigma$ and $\mathcal{H}^n$ otherwise.)

Since Γ_Σ is compact, we also have $\mathcal{H} \in \Gamma_\Sigma$.

Now we know that our inductively defined set $\mathcal{H}$ is indeed in Γ_Σ and that $H \subseteq \mathcal{H}$. It only remains to show that $\mathcal{H}$ is maximal in Γ_Σ. So let $\mathbf{A}_n \in \mathit{wff}_o(\Sigma)$ be the n-th sentence from the above sequence, such that $\mathcal{H} * \mathbf{A}_n \in \Gamma_\Sigma$. Since $\mathcal{H}$ is closed under subsets we know that $\mathcal{H}^n * \mathbf{A}_n \in \Gamma_\Sigma$. By definition of $\mathcal{H}^{n+1}$ we conclude that $\mathbf{A}_n \in \mathcal{H}^{n+1}$ and hence $\mathbf{A}_n \in \mathcal{H}$. □

Next we define two congruence relations which we need in the model existence theorems below in order to build quotient models.

Definition 4.25. Let Γ_Σ be an abstract consistency class and $\mathcal{H}$ be a Hintikka set for Γ_Σ. For all $\mathbf{A}, \mathbf{B} \in \mathit{wff}(\Sigma)$ we define:

$\mathbf{A}_\gamma \dot{\sim}_\mathcal{H} \mathbf{B}_\gamma$, iff $\mathbf{A} \doteq^\gamma \mathbf{B} \in \mathcal{H}$.

$$\mathbf{A}_\gamma \sim_\mathcal{H} \mathbf{B}_\gamma, \text{ iff } \begin{cases} \mathbf{A} \equiv \mathbf{B} & \text{if } \gamma \equiv \iota \\ \{\mathbf{A}, \mathbf{B}\} \in \mathcal{H} \text{ or } \{\mathbf{A}, \mathbf{B}\} \cap \mathcal{H} \equiv \emptyset & \text{if } \gamma \equiv o \\ \mathbf{AC} \sim_\mathcal{H} \mathbf{BC} \text{ for all } \mathbf{C} \in \mathit{wff}_\alpha(\Sigma) & \text{if } \gamma \equiv \alpha \to \beta \end{cases}$$

Lemma 4.26. *Let Γ_Σ be an abstract consistency class and $\mathcal{H}$ be a Hintikka set for Γ_Σ. Then $\sim_\mathcal{H}$ is a functional congruence relation, if Γ_Σ is an $\mathfrak{Acc}_{\mathfrak{M}}$ and $\dot{\sim}_\mathcal{H}$ is a functional congruence relation, if Γ_Σ is an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}$.*

Proof:

$\dot{\sim}_\mathcal{H}$ is a functional congruence relation by $\overline{\nabla}_{\doteq^r}$, $\overline{\nabla}_{\doteq^{sy}}$, $\overline{\nabla}_{\doteq^{tr}}$, $\overline{\nabla}_{\mathfrak{q}}$ and $\overline{\nabla}_{\mathfrak{q}'}$, which are valid in case Γ_Σ is an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}$.

Note that $\sim_\mathcal{H}$ is a functional congruence by construction. □

Remark 4.27. Note that in 3.36 $\mathrm{EXT}_L^{\alpha\to\beta}$ does not hold for $\doteq$ and hence $\dot{\sim}_\mathcal{H}$ is not a functional congruence in case Γ_Σ is not at least an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{q}}}$. Hence $\dot{\sim}_\mathcal{H}$ is unsuitable for the model construction of an $\mathfrak{M}_{\mathfrak{fb}}$ (or $\mathfrak{M}_{\mathfrak{b}}$) from a given $\mathfrak{M}_{\mathfrak{f}}$ (or $\mathfrak{M}$) as demonstrated below but fits well for the construction of an $\mathfrak{M}_{\mathfrak{qb}}$ or $\mathfrak{M}_{\mathfrak{qb}}$. Fortunately the relation $\sim_\mathcal{H}$ is already a functional congruence in case Γ_Σ is an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{f}}}$.

We now use the Σ-Hintikka sets, guaranteed by lemma 4.24, to construct a Σ-valuation for the Σ-term structure that turns it into the desired model $\mathcal{M}$.

Theorem 4.28 (Model Existence). *Let Γ_Σ be an saturated $\mathfrak{Acc}$ and $H \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences. For all $* \in \{\mathfrak{M}, \mathfrak{M}_{\mathfrak{f}}, \mathfrak{M}_{\mathfrak{fb}}, \mathfrak{M}_{\mathfrak{q}}, \mathfrak{M}_{\mathfrak{qb}}, \mathfrak{M}_{\mathfrak{e}}, \mathfrak{M}_{\mathfrak{eb}}\}$ we have: If Γ_Σ is an $\mathfrak{Acc}_*$, then there exists a countable model in * that satisfies H.*

Proof: Let Γ_Σ be an abstract consistency class. We can assume without loss of generality (see lemma 4.12) that Γ_Σ is compact, so the preconditions of 4.24 are met, and therefore there exists a Σ-Hintikka set $\mathcal{H} \subseteq \mathit{wff}_o(\Sigma)$ for Γ_Σ, such that $H \subseteq \mathcal{H}$.

Now, for each different kind of abstract consistency class, we will construct a countable model $\mathcal{M}^{\mathcal{H}}$ of the corresponding type. These model constructions closely reflect the relations of the different model types as discussed in section 3 and shown in figures 1 and 2. We start with the construction of a $\mathfrak{M}$ and a $\mathfrak{M}_\mathfrak{f}$ based upon the non-functional termstructure $\mathcal{TS}(\Sigma)^\beta$ and the functional $\mathcal{TS}(\Sigma)^{\beta\eta}$. The remaining model constructions are then based upon these two basic constructions.

$\mathfrak{M}$ Let Γ_Σ be an $\mathfrak{Acc}_\mathfrak{M}$. Given the Σ-Hintikka set $\mathcal{H}$ with $H \subseteq \mathcal{H}$ from above, we choose $\upsilon(\mathbf{C}) := \mathtt{T}$, iff $\mathbf{C} \in \mathcal{H}$. Note that we have $\upsilon(\mathbf{C}) := \mathtt{F}$, iff $\neg\mathbf{C} \in \mathcal{H}$ by $\overline{\nabla}_{c^b}$. By $\overline{\nabla}_\beta$ we know that υ is well-defined on $\mathit{cwff}(\Sigma)_{\downarrow_\beta}$ and by $\overline{\nabla}_{c^b}$, $\overline{\nabla}_{c^c}$ we have that υ is a total function on $\mathcal{D}_o^\beta$.
Furthermore by $\overline{\nabla}_{c^b}$, $\overline{\nabla}_{c^c}$, $\overline{\nabla}_\vee$ and $\overline{\nabla}_\forall$ we have that υ is a Σ-valuation of the Σ-term structure $\mathcal{TS}(\Sigma)^\beta$ and thus $\mathcal{M}^{\mathcal{H}} := (\mathcal{TS}(\Sigma)^\beta, \upsilon)$ is a Σ-model by construction. We have $\mathcal{M}^{\mathcal{H}} \models H$, since $H \subseteq \mathcal{H}$. Note that $\mathcal{M}^{\mathcal{H}}$ is indeed countable, since the sets of well-typed formulae are countable.

$\mathfrak{M}_\mathfrak{f}$ Let Γ_Σ be an $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{f}}$ and hence also an $\mathfrak{Acc}_\mathfrak{M}$. Analogous to the previous case we construct the countable Σ-model $\mathcal{M}^{\mathcal{H}} := (\mathcal{TS}(\Sigma)^{\beta\eta}, \upsilon)$ with $\mathcal{M} \models H$. Note that in this case υ is well-defined on $\mathcal{D}_T^{\beta\eta}$ because of $\overline{\nabla}_\mathfrak{f}$. By lemma 3.15 we know that $\mathcal{M}$ is functional and hence $\mathcal{M}$ is an $\mathfrak{M}_\mathfrak{f}$.

We proceed with the construction of a $\mathfrak{M}_\mathfrak{q}$ and $\mathfrak{M}_\mathfrak{b}$ based upon the previous construction of a $\mathfrak{M}$ and accordingly of a $\mathfrak{M}_\mathfrak{q}$ and a $\mathfrak{M}_{\mathfrak{fb}}$ based upon a $\mathfrak{M}_\mathfrak{f}$. Thus we start out with a countable Σ-model $\mathcal{M} := (\mathcal{TS}(\Sigma)^\beta, \upsilon)$ or $\mathcal{M} := (\mathcal{TS}(\Sigma)^{\beta\eta}, \upsilon)$ such that $\mathcal{M} \models H$. Property $\mathfrak{q}$ is easy to verify, as it follows from the properties discussed in the Hintikka-lemmata 4.14 and 4.16.

$\mathfrak{M}_\mathfrak{q}$ Let Γ_Σ be an $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{q}}$. From $\overline{\nabla}_{\dot{=}^r}$, $\overline{\nabla}_{\dot{=}^{sy}}$, $\overline{\nabla}_{\dot{=}^{tr}}$, $\overline{\nabla}_\mathfrak{q}$ and $\overline{\nabla}_{\mathfrak{q}'}$ we can derive that $\dot{=}^\alpha$ is indeed the q^α required by property $\mathfrak{q}$ and hence $\mathcal{M}$ is a countable $\mathfrak{M}_\mathfrak{q}$.

To verify property $\mathfrak{b}$ instead we have to construct a $\mathcal{M}'$ from $\mathcal{M}$ by reducing the set of truth values to $\{\mathtt{T}, \mathtt{F}\}$, which can be done with the help of a functional congruence relation.

$\mathfrak{M}_\mathfrak{b}$,$\mathfrak{M}_{\mathfrak{fb}}$ Let Γ_Σ be an $\mathfrak{Acc}_{\mathfrak{M}_\mathfrak{q}}$ or $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{fb}}}$. By lemma 4.26, $\overline{\nabla}_{\mathfrak{b}'}$ and $\overline{\nabla}_{c^c}$ we can show that the relation $\sim_\mathcal{H}$ defined in 4.25 is a functional Σ-congruence for $\mathcal{M}$ and thus, by lemma 3.33, the quotient structure $\mathcal{M}/_{\sim_\mathcal{H}}$ is a functional Σ-model which satisfies H. From $\overline{\nabla}_{c^b}$, $\overline{\nabla}_{c^c}$ and the choice of υ we conclude that $\sim_\mathcal{H}$ has exactly two equivalence classes on $\mathcal{TS}_o(\Sigma)^{\beta\eta}$. Thus we have $\mathcal{D}_o \equiv \{\mathtt{T} := [\![\mathbf{T}_o]\!], \mathtt{F} := [\![\mathbf{F}_o]\!]\}$, if we define $\mathbf{T}_o$ and $\mathbf{F}_o$ as in lemma 3.25. Using $\overline{\nabla}_t$ and $\overline{\nabla}_{c^b}$ we further get that υ is the identity relation. Finally note that $\mathcal{M}/_{\sim_\mathcal{H}}$ is countable since $\mathcal{M}$ is.

We finish the constructions for the cases without a primitive notion of equality with the construction of a Σ-Henkin model ($\mathfrak{H} = \mathfrak{M}_{\mathfrak{qb}}$) in case we are considering an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}$.

We start with the $\mathfrak{M}_\mathfrak{q}$ $\mathcal{M}$ guaranteed by the discussion above. Analogous to the construction of a $\mathfrak{M}_\mathfrak{b}$, we make use of a functional congruence relation in order to construct a quotient model which fulfills property $\mathfrak{b}$. But instead of the relation $\sim_\mathcal{H}$ we had to use before, we apply the simpler relation $\dot{\sim}_\mathcal{H}$ which is a functional congruence relation for $\mathfrak{M}_{\mathfrak{qb}}$s.

$\mathfrak{H} = \mathfrak{M}_{\mathfrak{qb}}$ By lemma 4.26 and $\overline{\nabla}_{\mathfrak{b}'}$,$\overline{\nabla}_{c^c}$ we know that the relation $\dot{\sim}_\mathcal{H}$ is a functional Σ-congruence for $\mathcal{M}$, so the quotient structure $\mathcal{M}/_{\dot{\sim}_\mathcal{H}}$ is a $\mathfrak{M}_\mathfrak{q}$ with $\mathcal{M}/_{\dot{\sim}_\mathcal{H}} \models H$ by lemma 3.33. From $\overline{\nabla}_{c^b}$, $\overline{\nabla}_{c^c}$ and the choice of υ, we conclude that $\sim_\mathcal{H}$ has exactly two equivalence classes on $\mathcal{TS}_o(\Sigma)^{\beta\eta}$. Thus we have $\mathcal{D}_o \equiv \{\mathtt{T} := [\![\mathbf{T}_o]\!], \mathtt{F} := [\![\mathbf{F}_o]\!]\}$, if we define $\mathbf{T}_o$ and $\mathbf{F}_o$ as in lemma 3.25. Using $\overline{\nabla}_t$ and $\overline{\nabla}_{c^b}$ we further get that υ is the identity relation. Finally note that $\mathcal{M}/_{\dot{\sim}_\mathcal{H}}$ is countable since $\mathcal{M}$ is.

It remains to discuss the cases with primitive equality and we start with the $\mathfrak{M}_\mathfrak{q}$, resp. $\mathfrak{M}_{\mathfrak{qb}}$ from above.

$\mathfrak{M}_{\mathfrak{e}}$ Let Γ_Σ be an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{e}}}$ and hence an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{q}}}$. We construct the countable $\mathfrak{M}_{\mathfrak{q}}$ $\mathcal{M}$ with $\mathcal{M} \models H$ as discussed above. It remains to show that property $\mathfrak{e}$ is valid for $\mathcal{M}$ which follows from property $\mathfrak{q}$ by $\overline{\nabla}_{=\doteq}$ and $\overline{\nabla}_{=\doteq'}$.

$\mathfrak{M}_{\mathfrak{eb}}$ This case is analogous: Let Γ_Σ be an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{eb}}}$ and hence an $\mathfrak{Acc}_{\mathfrak{M}_{\mathfrak{qb}}}$. We construct a countable $\mathfrak{H}$ $\mathcal{M} := (\mathcal{TS}(\Sigma)^{\beta\eta}, \upsilon)$ with $\mathcal{M} \models H$. It remains to show that property $\mathfrak{e}$ is valid for $\mathcal{M}^{\mathcal{H}}$ which follows again from property $\mathfrak{q}$ by $\overline{\nabla}_{=\doteq}$ and $\overline{\nabla}_{=\doteq'}$. □

4.5 Order Effects

It is common to stratify higher-order logics with respect to the complexity of symbols and types allowed to occur in formulae. We will use this stratification for a finer analysis of model existence for functional Σ-models in this section.

Traditionally, the orders of formulae is defined by the order of the types of symbols occurring in them: for any k, the formulae of $2k^{th}$ **order logic** are those in which no variable or parameter of order greater than k occurs and the formulae of $2k-1^{th}$ **order logic** are those in which no variable of order greater than k is quantified over. Here, the **order** $\mathbf{ord}^n(\alpha)$ of a type $\alpha \in \mathcal{T}$ is defined inductively as $\mathbf{ord}^n(\iota) = 0$ and $\mathbf{ord}^n(o) = 1$, and $\mathbf{ord}^n(\overline{\alpha_n} \to \beta) = \max_{i \leq n} \mathbf{ord}(\alpha_i) + 1$, where $\beta \in \mathcal{BT}$. With this convention, first-order logic is the classical notion, only individual variables can be quantified over.

In this paper, we will adopt a slightly different definition of order that does not distinguish between quantified variables or constants as a finer distinction does not seem to yield suitable restrictions for our model existence theorems. Moreover, we do not commit to a particular order, since can identify sufficient conditions a general order function μ.

Definition 4.29 (Type Ordering). We call a function $\mu : \mathcal{T} \longrightarrow \mathbb{N}$ a **type ordering**, if $\mu(\alpha), \mu(\beta) \leq \mu(\alpha \to \beta)$. We say that α is **of order** $k \in \mathbb{N}$, iff $\mu(\alpha) \leq k$. We will say that a formula is of order k, iff the types of all of its its subterms are. Similarly, for a signature Σ or a substitution σ, where we require all constants (all $\sigma(X)$, where $\sigma(X) \neq X$) to be of order k.

We will denote the sets of (closed) well-formed formulae of order k with $\mathit{wff}^k(\Sigma)$ ($\mathit{cwff}^k(\Sigma)$). Note that the sets $\mathit{wff}^k_\alpha(\Sigma)$, $\mathit{cwff}^k_\alpha(\Sigma)$ are empty if $k \leq \mu(\alpha)$, and that $\mathit{wff}^k(\Sigma)$, $\mathit{cwff}^k(\Sigma)$ are closed under substitutions of order k and is therefore also closed under β-reduction.

We will call a type ordering μ **finite**, iff for any given $k \in \mathbb{N}$, the set $\mathcal{T}^k_\mu := \{\alpha \mid \mu(\alpha) \preceq k, k \in \mathbb{N}\}$ is finite.

*Example 4.30 (**Type Ordering**).* We have already mentioned the classical ordering scheme for higher-order types above. Note that $\mathbf{ord}^n$ is a type ordering and if we define $\mathbf{ord}(\alpha \to \beta) = \max(\alpha, \beta) + 1$, then $\mathbf{ord}^n$ is a finite type ordering.

The function $\mu : \mathcal{T} \longrightarrow \{0\}$ that gives all types the order zero is trivially a type orderin, albeit a very uninteresting one, since with this ordering, all results of this section are subsumed by the results above.

With the notion of type ordering, we can make a finer distinction between Henkin and Standard model.

Definition 4.31 (k-Standard Models). Let μ be a type ordering, then we call a Σ-structure or Σ-model **k-standard wrt.** μ, iff $\mathcal{D}_{\alpha\to\beta}$ is full (i.e. if $\mathcal{D}_{\alpha\to\beta} = \mathcal{F}(\mathcal{D}_\alpha; \mathcal{D}_\beta)$) for all types $\alpha \to \beta$ of order $\geq k$. With this definition a standard model is a 0-standard one, since the order of functional type is positive. Clearly, we can construct from any Σ-structure or Σ-model $\mathcal{M}$ a k-standard one (which we will denote with $\mathcal{M}^k$), by replacing the function universes related to a type of order $\geq k$ appropriate full sets of functions and adjusting the application operator accordingly.

Definition 4.32 ($k - \Sigma$-Structure, $k - \Sigma$-Model). We call a pre-Σ-structure a $k-\Sigma$-structure, iff $\mathcal{E}_\varphi$ meets the conditions in 3.10 for all $\mathbf{A} \in \mathit{wff}^k(\Sigma)$, and we call a $k-\Sigma$-structure a $k-\Sigma$-model, iff it has a valuation.

Theorem 4.33 (Term Structure of order k).
Let $\mathcal{TS}(\Sigma)^{\beta\eta,k} := (\mathcal{D}^k, @^k, \mathcal{I}^k)$ be the k-standard Σ-structure induced by $\mathcal{TS}(\Sigma)^{\beta\eta}$, then $\mathcal{TS}(\Sigma)^{\beta\eta,k}$ is a functional Σ-structure, and furthermore for all $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$ that $\mathcal{I}^k_\varphi(\mathbf{A}) = \sigma(\mathbf{A})_{\downarrow_{\beta\eta}}$ if $\mathbf{ord}(\mathbf{A}) \leq k$ and $\sigma = \varphi|_{\mathbf{Free}(\mathbf{A})}$.

Proof: We prove that $\mathcal{I}^k_\varphi$ is a total function and that $\mathcal{I}^k_\varphi(\mathbf{A}) = \sigma(\mathbf{A})_{\downarrow_{\beta\eta}}$ if $\mathbf{ord}(\mathbf{A}) \leq k$ and $\sigma = \varphi|_{\mathbf{Free}(\mathbf{A})}$ by an induction on the structure of $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$. Note that totality of $\mathcal{I}^k_\varphi$ ensures that $\mathcal{TS}(\Sigma)^{\beta\eta,k}$ is a functional Σ-structure.

The case when $\mathbf{A} = c \in \Sigma$ is trivial, since totality of $\mathcal{I}^k_\varphi$ is guaranteed by the totality of $\mathcal{I}$ and if $\mathbf{ord}(\mathbf{A}) \leq k$ we have $\mathcal{I}^k_\varphi(\mathbf{A}) = \mathcal{I}(\mathbf{A}) = \mathbf{A} = \sigma(\mathbf{A})_{\downarrow_{\beta\eta}}$.

If $\mathbf{A} = X \in \mathcal{V}$, totality of $\mathcal{I}^k_\varphi$ follows from totality of Φ and furthermore in case $\mathbf{ord}(\mathbf{A}) \leq k$ we get since $\mathbf{X} \in \mathbf{Dom}(\sigma)$ that $\sigma(X)_{\downarrow_{\beta\eta}} = \varphi(X) = \mathcal{I}^k_\varphi(X)$.

Next we consider the case when $\mathbf{A}_\alpha$ is of form $\mathbf{B}_{\gamma\to\alpha}\mathbf{C}_\gamma$. If $\mathbf{ord}(\mathbf{A}) \leq k$ the assertions immediately follow from 3.15, since up to order k the construction of $\mathcal{TS}(\Sigma)^{\beta\eta,k}$ is identical with $\mathcal{TS}(\Sigma)^{\beta\eta}$. In case $\mathbf{ord}(A) > k$ note that $\mathcal{D}_\alpha$ is full and hence $\mathcal{I}^k_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha$, which is all we have to show.

In the last case $\mathbf{A}_\alpha$ is of form $\lambda X_\gamma.\mathbf{B}_\beta$. If $\mathbf{ord}(\mathbf{A}) > k$ we only have to show that $\mathcal{I}^k_\varphi(\mathbf{A}) \in \mathcal{D}_\alpha$ which holds since $\mathcal{D}_\alpha$ is full. If $\mathbf{ord}(\mathbf{A}) \leq k$ we have $\mathcal{I}^k_\varphi(\mathbf{A})@^k c = \mathcal{I}^k_{\varphi,[c/X]}(\mathbf{B})$ forall $c \in \mathcal{D}_\beta$ and by induction hypothesis that $\mathcal{I}^k_{\varphi,[c/X]}(\mathbf{B}) = (\sigma, [c/X](\mathbf{B}))_{\downarrow_{\beta\eta}}$. Furthermore we have $(\sigma, [c/X](\mathbf{B}))_{\downarrow_{\beta\eta}} = (\sigma([c/X]\mathbf{B}))_{\downarrow_{\beta\eta}} = (\sigma(Ac))_{\downarrow_{\beta\eta}} = (\sigma(A)c)_{\downarrow_{\beta\eta}} = (\sigma(A))_{\downarrow_{\beta\eta}}@^k c$. Thus $\mathcal{I}^k_\varphi(\mathbf{A}) = \sigma(A)_{\downarrow_{\beta\eta}}$. □

Now, we are in the position to prove a finer-grained model-existence theorem. For this we will first weaken the definition of an abstract consistency class by weakening the conditions for universal quantification in Definition 4.4 by restricting the sets possible instantiations.

Definition 4.34 (Abstract Consistency Class with Order). Let Γ^k_Σ be an $\mathfrak{Acc}_*$, where $* \in \{\mathfrak{M}, \mathfrak{M}_\mathfrak{b}, \mathfrak{M}_\mathfrak{f}, \mathfrak{M}_\mathfrak{fb}, \mathfrak{M}_\mathfrak{q}, \mathfrak{M}_\mathfrak{qb}, \mathfrak{M}_\mathfrak{e}, \mathfrak{M}_\mathfrak{eb}\}$ and the condition $\nabla_\forall$ is replaced by the following one

$\nabla^k_\forall$ If $\Pi^\alpha\mathbf{F} \in \Phi$ and $\mathbf{ord}(\Phi) = k$, then $\Phi * \mathbf{FG} \in \Gamma_\Sigma$ for each $\mathbf{G} \in \mathit{wff}^k_\alpha(\Sigma)$.

then we call Γ^k_Σ an $\mathfrak{Acc}_*$ **with order** k and write $\mathfrak{Acc}^k_*$.

Lemma 4.35 (k-Hintikka lemmata). *Clearly, we can prove analoga to all Hintikka theorems in section 4.2. The only difference lies in the $\overline{\nabla}_\forall$ and $\overline{\nabla}_\exists$ cases of Theorem 4.14 where we have*

$\overline{\nabla}_\forall$ $\Pi^\alpha\mathbf{F} \in \mathcal{H}$, *iff for each* $\mathbf{D} \in \mathit{cwff}^k_\alpha(\Sigma)$ *we have* $\mathbf{FD} \in \mathcal{H}$.

$\overline{\nabla}_\exists$ $\neg\Pi^\alpha\mathbf{F} \in \mathcal{H}$, *iff there is a* $\mathbf{D} \in \mathit{cwff}^k_\alpha(\Sigma)$ *such that* $\neg\mathbf{FD} \in \mathcal{H}$.

Proof: We get the first assertion directly by $\nabla^k_\forall$ and the second by $\nabla_\exists$ observing that the witness constant is of type α and therefore $\mu(\mathbf{D}) = \mu(\alpha) \leq \mu(\alpha \to o) = \mathbf{ord}(\Pi^\alpha) \leq \mu(\neg\Pi^\alpha\mathbf{F}) \leq \mu(\Phi)$. We have analogous assertions and argumentations for $\overline{\nabla}^k_\mathfrak{q}$ and $\overline{\nabla}^k_{\mathfrak{q}'}$ in Theorem 4.16 and for $\overline{\nabla}_{=^q}$ and $\overline{\nabla}_{=^{q'}}$ in Theorem 4.22. □

This gives us the following model existence theorem:

Theorem 4.36 (Model Existence). *Let Γ_Σ be a saturated $\mathfrak{Acc}^k$ and $H \in \Gamma_\Sigma$ be a sufficiently Σ-pure set of sentences. For all $* \in \{\mathfrak{M}_\mathfrak{f}, \mathfrak{M}_\mathfrak{fb}, \mathfrak{M}_\mathfrak{qb}, \mathfrak{M}_\mathfrak{e}, \mathfrak{M}_\mathfrak{eb}\}$ we have: If Γ_Σ is an $\mathfrak{Acc}^k_*$, then there exists a model in * that satisfies H.*

Proof: In the extension Lemma 4.24 we can guarantee a Γ_Σ-Hintikka set $\mathcal{H}$ with $\mathbf{ord}(\mathcal{H}) = \mathbf{ord}(H)$ for any set $H \in \Gamma_\Sigma$: If we take the sequence $\mathbf{A}^1, \mathbf{A}^2, \ldots$ to be an enumeration of $\mathit{wff}(\Sigma)^k$, where $k = \mu(H)$ and consider the construction of H^{n+1} in table 3, then we see that $\mu(H^{n+1}) = \mu(H^n)$, since the formulae added to H^n are either witness constants of type α or subformulae of $\mathbf{A}^n$. In both cases the order cannot be greater than k.

The constructions of the Σ-models are analogous to the respective constructions in Theorem 4.28. The only difference is that we will use $\mathcal{TS}(\Sigma)^{\beta\eta,k}$ instead of $\mathcal{TS}(\Sigma)^{\beta\eta}$ as a starting point. Since the Hintikka set $\mathcal{H}$ guaranteed by the extended extension Lemma has order k, and $\mathcal{TS}(\Sigma)^{\beta\eta,k}$ is a k-Σ-algebra by 4.33, the constructions from Theorem 4.28 go through directly. □

Remark 4.37. An application of this theorem is that we can use this strengthened theorem to prove the long-standing conjecture that that in machine-oriented calculi it is sufficient to restrict primitive substitutions [And89] to the order of the input formulae. This is important for the implementation of fair strategies in automated deduction systems, since the primitive substitution rule without this observation is infinitely branching (there are infinitely many quantifiers Π^α, since $\mathcal{T}$ is infinite). If we employ a finite type ordering μ, then we only have to consider the finite set of quantifiers Π^α, where $\alpha \in \mathcal{T}^k_\mu$. For practical implementations it remains to construct type orderings that make $\mathcal{T}^k_\mu$ as small as possible.

5 Conclusion

In this paper, we have given an overview over the landscape of semantics for classical higher-order logics. We have differentiated ten different possible notions and have tied the discerning properties to conditions of the abstract consistency classes. The model existence theorems presented in this paper can serve as an instrument for the design of higher-order calculi.

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