Cut-free Calculi for Challenge Logics in a Lazy Way*

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The development of cut-free calculi for expressive logics, e.g. quantified non-classical logics, is usually a non-trivial task. However, for a wide range of challenge logics there exists an elegant and uniform solution: By modeling and studying these logics as fragments of classical higher-order logic (HOL) [1, 4] — a research direction I have recently proposed [3] — existing results for HOL can often be reused. We illustrate the idea with quantified conditional logics [7].

HOL. Assuming a set of simple types $\mathcal{T}$, the syntax of HOL is given as $\mathbf{A}, \mathbf{B} ::= c_\alpha \mid X_\alpha \mid (\lambda X_\alpha \mathbf{A}_\beta)_{\alpha\to\beta} \mid (\mathbf{A}_{\alpha\to\beta}\, \mathbf{B}_\alpha)_\beta \mid (\neg_{o\to o}\, \mathbf{A}_o)_o \mid (\mathbf{A}_o \vee_{o\to o\to o} \mathbf{B}_o)_o \mid (\Pi_{(\alpha\to o)\to o}\, \mathbf{A}_{\alpha\to o})_o$, where $\alpha, \beta \in \mathcal{T}$, $o \in \mathcal{T}$ is the type of truth values $\{\top, \bot\}$, c_α are typed constant symbols declared in a signature Σ, X_α are typed variables, and $\neg$, $\vee$, and the Π^α are the primitive logical connectives (binder notation $\forall X_\alpha \mathbf{A}$ is used as an abbreviation for $\Pi_{(\alpha\to o)\to o} \lambda X_\alpha \mathbf{A}$). Further logical connectives and Leibniz equality $\doteq$ can be defined (e.g. $\mathbf{A}_\alpha \doteq \mathbf{B}_\alpha := \forall P_{\alpha\to o}(\neg P\mathbf{A} \vee P\mathbf{B})$). The semantics of HOL, here we assume Henkin semantics, is well understood; cf. [1, 4] and the references therein.

A Cut-free Sequent Calculus for HOL. Consider the following one-sided sequent calculus $\mathcal{G}_{\beta\mathfrak{fb}}$ [5] (Δ and Δ' are finite sets of β-normal closed formulas and $\Delta * \mathbf{A}$ denotes the set $\Delta \cup \{\mathbf{A}\}$, $cwff_\alpha(\Sigma)$ is the set of closed terms of type α, and $\mathbf{A}\!\downarrow_\beta$ denotes the β-normal form of $\mathbf{A}$):

Basic Rules

$$\frac{\mathbf{A} \text{ atomic (and } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg\mathbf{A}}\,\mathcal{G}(init) \qquad \frac{\Delta * \mathbf{A}}{\Delta * \neg\neg\mathbf{A}}\,\mathcal{G}(\neg) \qquad \frac{\Delta * \neg\mathbf{A} \quad \Delta * \neg\mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_-)$$

$$\frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_+) \qquad \frac{\Delta * \neg\,(\mathbf{A}\mathbf{C})\!\downarrow_\beta \quad \mathbf{C} \in cwff_\alpha(\Sigma)}{\Delta * \neg\Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^{\mathbf{C}}_-) \qquad \frac{\Delta * (\mathbf{A}c)\!\downarrow_\beta \quad c_\alpha \in \Sigma \text{ new}}{\Delta * \Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi^c_+)$$

Funct. and Boolean Extensionality

$$\frac{\Delta * (\forall X_\alpha . \mathbf{A}X \doteq^\beta \mathbf{B}X)\big\downarrow_\beta}{\Delta * (\mathbf{A} \doteq^{\alpha\to\beta} \mathbf{B})}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta * \neg\mathbf{A} * \mathbf{B} \quad \Delta * \neg\mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^o \mathbf{B})}\,\mathcal{G}(\mathfrak{b})$$

Initialization and Decomposition of Leibniz Equality

$$\frac{\Delta * (\mathbf{A} \doteq^o \mathbf{B}) \quad \mathbf{A},\mathbf{B} \text{ atomic}}{\Delta * \neg\mathbf{A} * \mathbf{B}}\,\mathcal{G}(Init^{\doteq})$$

$$\frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \ \cdots\ \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad n \ge 1, \beta \in \{o, \iota\}, h_{\overline{\alpha^n}\to\beta} \in \Sigma}{\Delta * (h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n})}\,\mathcal{G}(d)$$

Theorem 1 (cf. [5]): $\models^{\text{HOL}} \mathbf{A}_o$ iff $\vdash^{\mathcal{G}_{\beta\mathfrak{fb}}}_{\text{cut-free}} \mathbf{A}_o$

That is, $\mathcal{G}_{\beta\mathfrak{fb}}$ is cut-free, and sound and complete for HOL with Henkin semantics. While HOL usually considers base types i and o only, we assume three disjunct base types below. The type i is now associated with possible worlds and the additional type u stands for individuals. o still denotes the Booleans. This modification is non-critical and Theorem 1 remains valid.

Exemplary Challenge Logic: Quantified Conditional Logic. As an exemplary challenge logic we consider quantified conditional logic (QCL). The syntax of QCL is $\varphi, \psi ::= P \mid k(X^1, \ldots, X^n) \mid \neg\varphi \mid \varphi\vee\psi \mid \varphi \Rightarrow \psi \mid \forall^{co}X\varphi \mid \forall^{va}X\varphi \mid \forall P\varphi$, where the P resp. X^i are propositional resp. individual variables, and the symbols k are n-ary constants. $\Rightarrow$ represents the modal conditional operator, and $\forall^{co}X\varphi$ and $\forall^{va}X\varphi$ denote constant domain and varying domain quantification. Regarding semantics we assume selection function semantics [7]. QCLs have many many applications and, interestingly, they subsume normal modal logics ($\Box\varphi$ can be defined as $\neg\varphi \Rightarrow \varphi$).

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Modeling QCLs as fragments of HOL. QCL formulas φ can be identified with certain HOL terms (predicates) $\varphi_{i\rightarrow o}$ of type $i \rightarrow o$. The latter type is abbreviated as τ below. The HOL terms φ_τ can be applied to terms of type i, which are assumed to denote possible worlds. The core idea of the embedding is captured by the following set of equations, resp. set of HOL axioms, called **AX**:

$$\begin{aligned}
\neg_{\tau\rightarrow\tau} &= \lambda A_\tau \lambda X_i \neg(A\,X) \\
\vee_{\tau\rightarrow\tau\rightarrow\tau} &= \lambda A_\tau \lambda B_\tau \lambda X_i (A\,X \vee B\,X) \\
\Rightarrow_{\tau\rightarrow\tau\rightarrow\tau} &= \lambda A_\tau \lambda B_\tau \lambda X_i \forall V_i (f\,X\,A\,V \rightarrow B\,V) \\
\Pi^{co}_{(u\rightarrow\tau)\rightarrow\tau} &= \lambda Q_{u\rightarrow\tau} \lambda V_i \forall X_u (Q\,X\,V) \\
\Pi^{va}_{(u\rightarrow\tau)\rightarrow\tau} &= \lambda Q_{u\rightarrow\tau} \lambda V_i \forall X_u (eiw\,V\,X \rightarrow Q\,X\,V) \\
\Pi_{(\tau\rightarrow\tau)\rightarrow\tau} &= \lambda R_{\tau\rightarrow\tau} \lambda V_i \forall P_\tau (R\,P\,V)
\end{aligned}$$

$\neg_\tau$, $\vee_{\tau\rightarrow\tau\rightarrow\tau}$, $\Rightarrow_{\tau\rightarrow\tau\rightarrow\tau}$, $\Pi^{co,va}_{(u\rightarrow\tau)\rightarrow\tau}$ and $\Pi_{(\tau\rightarrow\tau)\rightarrow\tau}$ realize the QCL connectives as 'world-lifted' HOL terms encoding selection function semantics. As before, binder notation is avoided and appropriately defined Π-connectives in combination with λ-abstraction are used instead, e.g., QCL formula $\forall^{co}\varphi$ is associated with $\Pi^{co}_{(u\rightarrow\tau)\rightarrow\tau}\lambda X_u \varphi_\tau$. Constant symbol f in the mapping of $\Rightarrow$ is of type $i \rightarrow \tau \rightarrow \tau$. It realizes the selection function. Constant symbol eiw (for 'exists in world'), which is of type $(\tau \rightarrow u) \rightarrow \tau$, is associated with the varying domains. The interpretations of f and eiw are chosen appropriately in HOL. Moreover, propositional and individual QCL variables P and X are associated with corresponding HOL variables P_τ and X_u, and the n-ary QCL constant symbols k are identified with HOL constant symbols $k_{u^n\rightarrow\tau}$. For grounding the lifted formulas φ_τ, the notion of validity is introduced as $\mathrm{vld}_{\tau\rightarrow o} = \lambda A_\tau \forall S_i (A\,S)$. For the varying domains, the non-emptiness axiom $\forall W_i \exists X_u (eiw\,W\,X)$ is additionally postulated and included in the axiom set **AX**. Soundness and completeness of this embedding of QCL in HOL has been studied in [2].

Theorem 2 (cf. [2]): $\models^{QCL} \varphi$ iff $\mathbf{AX} \models^{\mathrm{HOL}} \mathrm{vld}\,\varphi_\tau$

Cut-free Sequent Calculus for QCL. Combining the above results we obtain a cut-free sequent calculus for QCL.

Theorem 3: $\models^{QCL} \varphi$ iff $\mathbf{AX} \vdash^{\mathcal{G}_{\beta\mathfrak{fb}}}_{\text{cut-free}} \mathrm{vld}\,\varphi_\tau$

However, we need to point to the subtle issue of cut-simulation (effective simulation of the cut rule in a cut-free calculus) in HOL, cf. [5] for details. For example, if the equations in **AX** are formalized as Leibniz equations, then cut-simulation applies and cut-freeness of $\mathcal{G}_{\beta\mathfrak{fb}}$ is practically worthless. To avoid the problem, the calculus $\mathcal{G}_{\beta\mathfrak{fb}}$ can be extended for primitive equality, cf. [6], and primitive equality can then be used for stating the axioms.

When postulating additional axioms for the embedded logics in HOL (e.g. for QCL axiom ID: $\forall\varphi(\varphi \Rightarrow \varphi)$), cut-simulation may nevertheless apply. In some cases, however, the semantical conditions which correspond to such cut-strong axioms can be postulated instead in order to circumvent the effect. This is e.g. possible for many modal logic axioms, but it does not apply to ID since the semantical condition that corresponds to ID, $\forall A_\tau \forall W_i (f\,W\,A \subseteq A)$, is still cut-strong.

References

1. P.B. Andrews. Church's type theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Stanford University, spring 2009 edition, 2009.
2. C. Benzmüller. Automating quantified conditional logics in HOL. In *23rd International Joint Conference on Artificial Intelligence (IJCAI 2013)*, Beijing, China, 2013.
3. C. Benzmüller. A top-down approach to combining logics. In *Proc. of the 5th International Conference on Agents and Artificial Intelligence (ICAART)*, Barcelona, Spain, 2013. SciTePress Digital Library.
4. C. Benzmüller, C.E. Brown, and M. Kohlhase. Higher-order semantics and extensionality. *Journal of Symbolic Logic*, 69(4):1027–1088, 2004.
5. C. Benzmüller, C.E. Brown, and M. Kohlhase. Cut-simulation and impredicativity. *Logical Methods in Computer Science*, 5(1:6):1–21, 2009.
6. C.E. Brown. *Reasoning in Simple Type Thoery.* College Publications, 2008.
7. R.C. Stalnaker. A theory of conditionals. In *Studies in Logical Theory*, pages 98–112. Blackwell, 1968.